Dimension and entropy in compact topological group*

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Dedicated to the memory of Professor Adalberto Orsatti, *il Maestro*

Abstract

We study the topological entropy $h(f)$ of endomorphisms f of compact-like groups. More specifically, the e-spectrum $\mathbf{E}_{top}(K)$ for a compact-like group K (namely, the set of all values $h(f)$, when f runs over $End(K)$), paying particular attention to the class $\mathfrak{E}_{<\infty}$ of groups without endomorphisms of infinite entropy (i.e., $\infty \notin \mathbf{E}_{top}(K)$) as well as the subclass $\mathfrak{E}_0$ of $\mathfrak{E}_{<\infty}$ consisting of those groups K with $\mathbf{E}_{top}(K) = \{0\}$. It turns out that the properties of the e-spectrum and these two classes are very closely related to the topological dimension. We show, among others, that a compact connected group G with finite-dimensional commutator subgroup belongs to $\mathfrak{E}_{<\infty}$ if and only if $\dim G < \infty$ and we obtain a simple formula (involving the entropy function) for the dimension of an abelian topological group which is either locally compact or ω-bounded (in particular, compact). Examples are provided to show the necessity of the compactness or commutativity conditions imposed for the validity of these results (e.g., compact connected semi-simple groups G with $\dim G = \infty$ and $G \in \mathfrak{E}_0$, or countably compact connected abelian groups with the same property). Since the class $\mathfrak{E}_{<\infty}$ is not stable under taking closed subgroups or quotients, we study also the largest subclasses $\mathsf{S}(\mathfrak{E}_{<\infty})$ and $\mathsf{Q}(\mathfrak{E}_{<\infty})$, respectively, of $\mathfrak{E}_{<\infty}$, having these stability properties. We provide a complete description of these two classes in the case of compact groups, that are either abelian or connected. The counterpart for $\mathsf{S}(\mathfrak{E}_0)$ and $\mathsf{Q}(\mathfrak{E}_0)$ is done as well.

1 Introduction

Inspired by Kolmogorov-Sinai's notion of measure-theoretic entropy, Adler, Konheim and McAndrew [1] introduced the concept of topological entropy for continuous self-maps of compact topological spaces. Then it was generalized by Bowen [5] for uniformly continuous self-maps of metric spaces. Hood [18] extended Bowen's entropy in the more general context of uniform spaces. In this framework, the study of Bowen's entropy in the realm of topological groups is justified and motivated. Recall that, for a surjective endomorphism on a compact group, the topological entropy coincides with the Kolmogorov-Sinai entropy with respect to the normalized Haar measure on the group ([3, 31]). The Kolmogorov-Sinai

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entropy appears in several interesting situations; for example, under certain conditions, it is the rate of increase of the statical entropy of a simple, fully chaotic conservative system [21].

It was shown by Stoyanov [31] that the entropy of continuous endomorphisms of compact groups is uniquely determined by a set of seven natural axioms satisfied by the topological entropy. Axiomatic definitions of entropy have been studied in different settings by several authors. For instance, Rokhlin [30] axiomatically defined the measure-theoretic entropy of an automorphism of a Lebesgue space, and Llibre and Snoha [25] gave two axiomatic definitions for the entropy of continuous self-maps on a compact interval.

The topological entropy, in the above sense, will be denoted by h in the sequel. Some of the most outstanding problems in discrete dynamical systems deal with the values of entropy, for instance, the existence of group automorphisms on compact groups with small entropy. One of the most celebrated can be written out as follows: taking the infimum over all compact group automorphisms α, is

$$\inf\{\, h(\alpha) \,|\, h(\alpha) > 0 \,\} > 0\,? \tag{1}$$

It is shown in [23] that a negative answer is equivalent to the statement that for any $a \in]0, +\infty]$ there is an (ergodic) compact group automorphism with topological entropy a (see also [10] for other equivalent forms of (1) involving the algebraic entropy). On the other hand, a positive answer implies that the set of all possible values of topological entropies of compact group automorphisms is countable (for a combinatorial analogue to (1) the reader might see [35] and for its relationship with Lehmer's Problem one can consult [22, 23, 10]).

1.1 Main results

This paper's predecessor [2] studied the infinitude of Bowen's entropy by providing a wealth of zero entropy endomorphisms on a totally bounded topological group whose extension to the (compact) Weil completion of the group has infinite entropy.

This paper deals with natural questions related to the values of Bowen's entropy in the framework of (mostly, compact) group endomorphisms. To this end we study the e-spectrum $\mathbf{E}_{top}(K) = \{h(f) : f \in End(K)\}$ for a compact group K. Our first aim is to study the class $\mathfrak{E}_\infty$ of topological groups K without endomorphisms of infinite entropy (i.e., such that $\infty \notin \mathbf{E}_{top}(K)$), as well as the class $\mathfrak{E}_0$ of topological groups K with the property $\mathbf{E}_{top}(K) = \{0\}$ (i.e., such that every endomorphism of K has zero entropy). Obviously $\mathfrak{E}_{<\infty} \supseteq \mathfrak{E}_0$. This inclusion is proper, according to Corollary 1.

It turns out that $\mathfrak{E}_0$ contains the remarkable class $\mathfrak{O}$ of Orsatti groups (Proposition 3.4), introduced by Orsatti in equivalent terms in a different setting [26]:

Definition 1.1. ([7]) A group of the form $G = \prod_{p\in\mathbb{P}} \mathbb{Z}_p^{n_p} \times F_p$, where $\mathbb{Z}_p$ is the compact group of p-adic integers, $n_p \geq 0$ is an integer and F_p is a finite p-group for every prime p, is called an *Orsatti group*.

In a preliminary draft of this paper the problem of whether every compact abelian group in $\mathfrak{E}_0$ is an Orsatti group was raised. By means of Pontryagin duality and Theorem 3.15, this problem was formulated also as a problem about torsion abelian groups and the algebraic entropy of their endomorphisms. This *dual* algebraic problem triggered the papers [10, 11] (see also [9, 14, 33], as well as the survey [8] and the references given there) dedicated to the algebraic entropy. Making use of the family built in [11, Theorem 5.4], we provide here a plenty of examples of compact abelian groups $G \in \mathfrak{E}_0$ that are not Orsatti groups (see Theorem E (b)). Nevertheless, the class $\mathfrak{O}$ allows for natural characterizations via the class $\mathfrak{E}_0$ (see Theorem F (a_2)).

Our first three theorems (as well as Theorem F) reveal a deep connection between the classes $\mathfrak{E}_{<\infty}, \mathfrak{E}_0$ (restricted to compact-like groups) and the topological dimension. We need to recall first, that a topological group G is *ω-bounded* if every countable subset of G is contained in some compact subset of G.

For a topological abelian group G and for every integer k we denote by m_k^G the continuous endomorphism $G \to G$ defined by the multiplication by k, i.e., $m_k^G(x) = kx$ for every $x \in G$. The next theorem shows that the entropy of the endomorphism m_k^G of a compact abelian group G can "measure" the dimension of G in a rather natural way (see (2)). In particular, the compact-like abelian groups in $\mathfrak{E}_{<\infty}$ are finite-dimensional.

Theorem A. *Let G be a topological abelian group that is either locally compact or ω-bounded. Then*

$$h(m_k^G) = \dim G \cdot \log k \tag{2}$$

for every integer $k > 1$. Moreover, the following are equivalent:

(a) G is totally disconnected;

(b) $h(m_k^G) = 0$ for every integer k;

(c) $h(m_k^G) = 0$ for some integer $k > 1$.

In particular, if $G \in \mathfrak{E}_{<\infty}$, then $\dim G < \infty$.

We show that the level of compactness cannot be lowered to countable compactness (see Example 4.2). On the other hand, in the compact case a more precise connection is available:

Theorem B. *Let G be a compact abelian group. Then :*

(a) if $\dim G < \infty$ and $G/c(G) \in \mathfrak{E}_{<\infty}$, then also $G \in \mathfrak{E}_{<\infty}$.

(b) $G \in \mathfrak{E}_0$ iff G is totally disconnected and $G = \prod_p G_p$, where $G_p \in \mathfrak{E}_0$ is a pro-p-group for every prime p.

Obviously, (a) is only a partial inverse of the last assertion of Theorem A. On the other hand, we do not know whether the condition $G/c(G) \in \mathfrak{E}_{<\infty}$ in (a) is necessary (see Question 7.3). For connected group one obviously has this bold equivalence:

Corollary 1. *A connected compact abelian groups belongs to $\mathfrak{E}_{<\infty}$ if and only of $\dim G < \infty$.*

The subclass $\mathfrak{E}_0$ of $\mathfrak{E}_{<\infty}$ is proper, since every compact abelian group $G \in \mathfrak{E}_0$ is totally disconnected. In contrast with this, the next example shows that $\mathfrak{E}_0$ may contain non-abelian compact connected groups (compare to item (b) of Theorem B):

Example 1.2. Let $K = SO_3(\mathbb{R})$. Then K is compact, connected, with $\dim K = 3$ and every non-trivial endomorphism ψ of K is an internal automorphism, so $h(\psi) = 0$ and $K \in \mathfrak{E}_0$ (see Remark 2.5 (a) and Example 6.8 for more a proof and general results in this direction).

In order to show that $G \in \mathfrak{E}_{<\infty}$ does not imply $\dim G < \infty$ in the non-abelian case, we provide in Example 6.8 a connected group $G = G' \in \mathfrak{E}_0$ with $\dim G = \infty$. The next theorem shows that even a rather mild commutativity condition (as $\dim G' < \infty$) may repair this problem. Item (a) reinforces Corollary 1:

Theorem C. *Let G be a compact connected group with $\dim G' < \infty$. Then :*

(a) $G \in \mathfrak{E}_{<\infty}$ if and only if $\dim G < \infty$;

(b) the following are equivalent:

(b_1) $G \in \mathfrak{E}_0$;

(b_2) $G = G'$;

(b_3) $\dim Z(G) = 0$; and

(b_4) $Z(G)$ is finite.

In particular, (a) and (b) hold for every compact connected abelian group G.

Theorems A will be proved in §4.1, Theorems B and C – in §4.2.

Combining Theorems A, B and C we can see that for a non-trivial finite-dimensional compact connected abelian group G, say with $0 < d := \dim G < \infty$, one has

$$\{k \log d : k \in \mathbb{N}\} \subseteq \mathbf{E}_{top}(G) \subseteq (0, +\infty).$$

In particular, $\mathbf{E}_{top}(G) = \{0, \infty\}$ can never occur for such a group G (see Example 7.1 for a compact connected (necessarily infinite-dimensional) abelian group G with $\mathbf{E}_{top}(G) = \{0, \infty\}$).

While $\mathbf{E}_{top}(G) = \{0\}$ (i.e., $G \in \mathfrak{E}_0$) is equivalent to $G = \{0\}$ for a compact connected abelian group G, the class $\mathfrak{E}_0$ may contain a plenty of totally disconnected compact abelian groups (actually, all totally disconnected compact abelian groups in $\mathfrak{E}_{<\infty}$ are already in the smaller class $\mathfrak{E}_0$, see Corollary 2). This means that for a totally disconnected compact abelian group K one has the dichotomy

$$\text{either } \mathbf{E}_{top}(K) = \{0\} \text{ (i.e., } K \in \mathfrak{E}_0) \quad \text{or} \quad \{0, \infty\} \subseteq \mathbf{E}_{top}(K).$$

In other words, we obtain an "intermediate smallness property" $\mathbf{E}_{top}(K) = \{0, \infty\}$ for the topological e-spectrum of totally disconnected compact abelian group K. More precisely, the next theorem shows that the existence of at least one finite positive value in $\mathbf{E}_{top}(G)$ yields $\infty \in \mathbf{E}_{top}(G)$, i.e., $G \notin \mathfrak{E}_{<\infty}$. Items (b) and (c) of the theorem describe the structure of the totally disconnected compact abelian groups K satisfying $\mathbf{E}_{top}(K) \not\subseteq \{0, \infty\}$ (for the term semi-standard see Definition 2.3):

Theorem D. *For an infinite totally disconnected compact abelian group K the following conditions are equivalent:*

(a) $\mathbf{E}_{top}(K) \not\subseteq \{0, \infty\}$;

(b) the group K is not semi-standard;

(c) K has a direct summand of the form $\mathbb{Z}(p^n)^{\mathbb{N}}$ for some $n \in \mathbb{N}$ and some prime p, where $\mathbb{Z}(p^n)$ is the cyclic group of order p^n;

(d) $\mathbf{E}_{top}(K) \supseteq \{\infty\} \cup \{mn \log p : m \in \mathbb{N}\}$ for some prime p and $n \in \mathbb{N}$;

This theorem will be proved in §5.1. The proof heavily relies on the Bridge Theorem 3.15 and properties of the algebraic entropy ent established in [11]. It implies that $K \notin \mathfrak{E}_0$ if and only if $\{0,\infty\} \subseteq \mathbf{E}_{top}(K))$ (i.e., $G \notin \mathfrak{E}_{<\infty}$) for an infinite totally disconnected compact abelian group K:

Corollary 2. *The totally disconnected compact abelian groups in $\mathfrak{E}_0$ and $\mathfrak{E}_{<\infty}$ coincide. In particular, they are semi-standard*

Proof. If $G \notin \mathfrak{E}_0$ for a totally disconnected compact abelian group G, then there exists endomorphism f of G with $h(f) > 0$. If $h(f) = \infty$, then $G \notin \mathfrak{E}_{<\infty}$. If $0 < h(f) < \infty$, then certainly $\mathbf{E}_{top}(G) \neq \{0,\infty\}$, so Theorem D (d) implies $G \notin \mathfrak{E}_{<\infty}$. □

Example 1.3. By the above corollary, every totally disconnected compact abelian groups in $\mathfrak{E}_0$ is semi-standard. More precisely, the semi-standard totally disconnected compact abelian groups G are precisely those with $\mathbf{E}_{top}(K) \subseteq \{0,\infty\}$. According to item (d) of Theorem D, a typical example of a totally disconnected compact abelian group satisfying $\mathbf{E}_{top}(K) = \{0,\infty\}$ is the group $K = \prod_{n=1}^{\infty} \mathbb{Z}(p^n)$ for a prime p. More generally, for a sequence $\{k_n\}_{n\in\mathbb{N}}$ of natural numbers, the group $K = \prod_{n=1}^{\infty} \mathbb{Z}(p^n)^{k_n}$ satisfies $\mathbf{E}_{top}(K) = \{0,\infty\}$ precisely when the sequence (k_n) is finitely many-to-one, i.e., when K is semi-standard.

Another typical example is a group K with $\mathbf{E}_{top}(K) = \{0,\infty\}$ (so, $K \notin \mathfrak{E}_{<\infty}$) is any group of the form $K = \prod_{p\in\mathbb{P}} \mathbb{Z}_p^{\kappa_p}$ with at least one infinite κ_p (e.g., every torsion-free totally disconnected compact abelian group of uncountable weight will do).

In item (a) of the next theorem we prove that $w(G) \leq \mathfrak{c}$ for compact totally disconnected abelian groups $\mathfrak{E}_{<\infty}$ ("totally disconnected" can be omitted in case Question 7.3 has a positive answer). In item (b) we show that the weight $\mathfrak{c}$ can be attained in $\mathfrak{E}_{<\infty}$ in many ways.

Theorem E.

(a) If $K \in \mathfrak{E}_{<\infty}$ is a totally disconnected compact abelian group, then $w(K) \leq \mathfrak{c}$.

(b) There is a family of $2^{\mathfrak{c}}$ many pairwise non-isomorphic compact groups $\{G_i : i \in I\}$ of weight $\mathfrak{c}$ such that

(b_1) $(G_i, \tau) \in \mathfrak{E}_0$ for each i;

(b_2) the torsion subgroup $t(G_i)$ is dense in G_i for each $i \in I$.

Item (b) of the above theorem will be deduced from results obtained in [11] and the Bridge Theorem 3.15. As far as item (b_2) is concerned, let us note that if K is a torsion-free totally disconnected compact abelian group with $w(K) = \mathfrak{c}$, then $K \notin \mathfrak{E}_{<\infty}$ (see Example 1.3).

The classes $\mathfrak{E}_{<\infty}$ and $\mathfrak{E}_0$ are not closed with respect to taking quotients and same applies to stability with respect to taking closed subgroups. Denote by $\mathsf{S}(\mathfrak{E}_{<\infty})$ (resp., by $\mathsf{Q}(\mathfrak{E}_{<\infty})$) the class of topological groups G such that all closed subgroup (all Hausdorff quotient, resp.) of G belongs to $\mathfrak{E}_{<\infty}$. Obviously, $\mathsf{S}(\mathfrak{E}_{<\infty})$ (resp., $\mathsf{Q}(\mathfrak{E}_{<\infty})$) is the largest subclass of $\mathfrak{E}_{<\infty}$ closed under taking closed subgroups (resp., quotients).

Theorem F. *Let G be a compact group.*

(a) If G is abelian, then

(a$_1$) $G \in \mathsf{S}(\mathfrak{E}_{<\infty})$ if and only if $\mathsf{Q}(\mathfrak{E}_{<\infty})$ if and only if $\dim G < \infty$ and $G/c(G) \in \mathfrak{O}$.

(a$_2$) $G \in \mathsf{S}(\mathfrak{E}_0)$ if and only if $G \in \mathsf{Q}(\mathfrak{E}_0)$ if and only if $G \in \mathfrak{O}$.

(b) Suppose G is connected.

(b$_1$) $G \in \mathsf{S}(\mathfrak{E}_{<\infty})$ if and only if $\dim G < \infty$. In such a case $G \in \mathsf{Q}(\mathfrak{E}_{<\infty})$ and G is metrizable.

(b$_2$) $G \in \mathsf{Q}(\mathfrak{E}_{<\infty})$ if and only if $\dim Z(G) < \infty$ and G has no Lie components of infinite multiplicity. In such a case, G is metrizable.

Theorem F is proved in §6, where we prove actually a more precise results regarding item (a) (see Proposition 6.2 and Corollary 6.6 for some more equivalent condition to those of items (a$_1$) and (a$_2$)).

The paper is organised as follows. Section 2 contains some necessary background on (topological) abelian groups and connected compact groups. Section 3 recalls some general properties on the topological entropy of group endomorphisms of (compact-like) topological groups, the algebraic entropy in discrete abelian groups and its connection to the topological one. We prove here also Proposition 3.4, showing that $\mathfrak{E}_0$ contains the class $\mathfrak{O}$. In Section 4 we prove Theorems A, B and C, while §5 provides examples of groups with endomorphisms of infinite entropy and the proofs of Theorems D and E using these examples.

In §6 we study the class $\mathfrak{O}_c$ of compact abelian group obtained by extending finite-dimensional compact connected abelian groups by groups from the class $\mathfrak{O}$. We show that the class $\mathfrak{O}_c$ is stable with respect to taking extensions, closed subgroups and quotients. Using these facts it is possible to see that these are exactly the groups described in item (a$_1$) of Theorem F. In §6.2 the case of connected non-abelian case is considered, here we prove item (b) of Theorem F.

The final section 7 provides some comments, conjectures and open questions.

We dedicate this paper to Adalberto Orsatti's 80th birthday, for his memorable work in the theory of abstract and topological abelian groups.

Notation and terminology

Our terminology and notation are standard. For instance, $\mathbb{P}$, $\mathbb{R}$ and $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ stand for the prime numbers, the additive group of the reals and the circle, respectively; $\mathbb{Z}(n)$ denotes the cycle group of order n and $\mathfrak{c}$ the continuum.

For an abelian group G and $m \in \mathbb{N}$ let

$$mG = \{mx : x \in G\},\ G[m] = \{x \in G : mx = 0\}, \quad \text{and} \quad t(G) := \bigcup_{n=1}^{\infty} G[n!],$$

the torsion subgroup of G. The group G is *divisible*, if $mG = G$ for all $m > 0$ The free rank of an abelian group G is denoted by $r(G)$. For $p \in \mathbb{P}$ the subgroup $G[p]$ is a vector space over the finite field $\mathbb{Z}/p\mathbb{Z}$, its dimension $r_p(G)$ over $\mathbb{Z}/p\mathbb{Z}$ is called *p-rank* of G. A subgroup H of G is *pure*, if $mG \cap H = mH$ for all $m \in \mathbb{N}$. For $p \in \mathbb{P}$ we denote by $\mathbb{Z}_p$ is the group of p-adic integers.

In the sequel, all topological groups will be assumed to be Hausdorff. Given a topological group G, we always consider it with its two-sided uniformity $\mathcal{U}$ uniformity. Then a continuous homomorphism from a topological group G_1 into a topological group G_2 is uniformly continuous respect to uniformities on G_1 and G_2. A topological group G is said to be *totally bounded* if for each neighbourhood U of the identity there exists a finite set $F \subset G$ such that $G = UF$, and with $H \leq G$ we denote a subgroup H of G.

For a topological group G we denote by $c(G)$ the connected component of G and by $w(G)$ the weight of G. We say that a topological group (G, τ) has a *linear topology* , if τ has a local base at 0 consisting of open subgroup. The Raĭkov completion of a topological group G will be denoted by $\widetilde{G}$, while the extension of a continuous endomorphism α of G to $\widetilde{G}$ is denoted by $\widetilde{\alpha}$. In the sequel we denote by $\widehat{G}$ the Pontryagin dual of a locally compact Abelian group G. In particular, $\mathbb{K}$ will denote the compact Pontryagin dual $\widehat{\mathbb{Q}}$ of the discrete group $\mathbb{Q}$ of the rational numbers. We denote by $\dim G$ the dimension of G. According to Pasynkov's celebrated theorem the three major dimension functions coincide for locally compact topological groups [27]. When G is compact abelian, $\dim G = r(\widehat{G})$.

For notions and terminology not defined here on topological groups the reader can consult [19], and on abstract abelian groups [16].

2 Background on abelian groups and entropy

2.1 p-groups and pro-p-groups

Every infinite abelian p-group X admits a *p-basic subgroup* $B \cong \bigoplus_{n=1}^{\infty} \mathbb{Z}(p^n)^{(f_n)}$, i.e., such that X/B is divisible and B is a pure subgroup of X (cf. [16]). The cardinals f_n, known as *Ulm-Kaplanski* invariants of X, are computed by the formula $f_n = r_p(p^{n-1}G[p]/p^nG[p])$. An abelian p-group G is called *semi-standard*, if all Ulm-Kaplanski invariants f_n of G are finite.

Let $B_m = \bigoplus_{n=1}^{m} \mathbb{Z}(p^n)^{(f_n)}$. The following fact is well-known:

In the sequel we intensively use pro-finite abelian group. Let us recall that these are inverse limits of finite abelian groups, and so are compact totally disconnected groups. The *pro-p-groups*, where p is a prime, are inverse limits of finite p-groups. In the next lemma we collect several properties of these groups used in the sequel.

Lemma 2.1. *Let p be a prime and let G be a pro-p-group. Then*

(a) the dual of G is a p-group;

(b) G is torsion-free if and only if $\widehat{G}$ is divisible if and only if $G \cong \mathbb{Z}_p^\kappa$ for some cardinal κ;

(c) G is torsion iff G has a finite exponent;

(d) If the quotient group G/N is torsion-free, then N splits topologically in G, i.e., $G = N \times H$ for some closed subgroup H of G isomorphic to G/N.

A pro-finite abelian group G is a topological direct product of pro-p-groups $G = \prod_{p\in\mathbb{P}} G_p$.

Proof. Items (a)–(c) are folklore (see [19, 12]).

(d) According to (b), the subgroup $N^\perp \cong \widehat{G/N}$ of $X = \widehat{G}$ is divisible, as the group G/N is torsion-free. Hence, $X = N^\perp \times Y$ for appropriate subgroup Y of X. Obviously, $Y \cong \widehat{G}/N^\perp \cong \widehat{N}$. Therefore, $X = N^\perp \times \widehat{N} = \widehat{G/N} \times \widehat{N} \cong \widehat{N \times G/N}$. Therefore, $G \cong \widehat{\widehat{G}} \cong N \times G/N$, i.e., N splits topologically in G.

For the last assertion take as the desired pro-p-subgroup G_p of G the set of all topologically p-torsion elements of G (i.e., the elements $x \in G$ such that $p^n x \to 0$, [4]). This gives the desired presentation (see [4, 12] for more details). $\square$

2.2 Dimension and connectedness of compact groups

Fact 2.2. *Let G be a locally compact group.*

(a) If N is a closed normal subgroup of the group G, then $\dim G = \dim G/N + \dim N$. *In particular,* $\dim G = \dim c(G)$.

(b) If G is totally disconnected, then G zero-dimensional (actually, G has linear topology, by a theorem of van Dantzig [19]).

(c) If G is totally disconnected and compact, then it is pro-finite, so can be described as in Lemma 2.1 in the abelian case.

(d) If G is compact abelian with $d = \dim G$, then there exists a continuous surjective homomorphism $f : G \to \mathbb{T}^d$ such that $\ker f$ *is totally disconnected.*

Definition 2.3. Call a totally disconnected compact abelian group G *semi-standard*, if for every prime p the p-group $\widehat{G_p}$ is semi-standard, where G_p is the pro-p-component of G (as in Lemma 2.1)

Here and in the sequel, the term *simple* used for a compact Lie group G means that G has no proper connected normal subgroups (but G may have non-trivial finite normal subgroups, e.g., a non-trivial finite center). When G has no proper normal subgroups at all, we say that G is *algebraically simple* or *centre-free.*

According to Goto's Theorem [20, Theorem 9.2], the commutator subgroup G' of a compact connected group G is closed, connected and every element of G' is a commutator. Following [20, Definition 9.5], call a compact connected group G *semisimple*, if $G = G'$.

Fact 2.4. *Let G be a compact connected group and let $A = c(Z(G))$.*

(a) [20, Corollary 9.20] *If G is semi-simple, then there is a family $\{S_i : i \in I\}$ of simple and simply connected compact connected Lie groups and a closed central subgroup N of $L = \prod_{i\in I} S_i$ such that $G \cong L/N$ and $Z(G) = Z(L)/N$.*

(b) [20, Theorem 9.24] *The closed subgroup $A \cap G'$ of G is totally disconnected and there is a family $\{S_i : i \in I\}$ as in item (a), such that with L as in item (a), there exists a closed totally disconnected subgroup N of $A \times Z(L)$ with $N \cap (A \times \{1\}) = \{1\}$ and $G \cong (A \times L)/N$; in particular, $G = A \cdot G'$. Moreover, $G/Z(G) \cong \prod_{i\in I} S_i/Z(S_i)$ has trivial center.*

Furthermore, $\dim G' < \infty$ *if and only if G' is a (connected semi-simple) Lie group, while*

$$\dim G = \dim Z(G) + \dim G' \quad \textit{and} \quad \dim Z(G) = \dim G/G'. \tag{3}$$

Consequently, G is metrizable when $\dim G < \infty$.

Remark 2.5. A few words are in order here related to the above fact.

(a) One can obtain the family $\{S_i : i \in I\}$ in the above fact by simply taking the quotient $G/Z(G)$ which is isomorphic to a product $\prod_{i\in I} L_i$ of compact connected algebraically simple Lie groups L_i. Now the covering groups $S_i = \widetilde{L}_i$ form the desired family.

(b) We refer to the Lie groups L_i appearing in the product $G/Z(G) \cong \prod_{i\in I} L_i$ by simply calling them *Lie components* of G (although $L_i \cong S/Z(S_i)$ is not necessarily isomorphic to a subgroup of G). There are only countably many pairwise non-isomorphic compact connected simple Lie groups. Therefore, in the product $G/Z(G) \cong \prod_{i\in I} L_i$ there are at most countably many pairwise non-isomorphic groups. If $\{\mathbb{L}_n : n \in \mathbb{N}\}$ is a complete list of all representatives of the isomorphism classes of the compact connected algebraically simple Lie groups, then one can rewrite the product (up to isomorphism) also in the form $\prod_{n=1}^{\infty} \mathbb{L}_n^{\kappa_n}$. We call the cardinal number κ_n *multiplicity* of the Lie component $\mathbb{L}_n$.

In the next lemma we show that every compact finite-dimensional group G contains a *largest* connected Lie subgroup $\lambda(G)$ (i.e., $\lambda(G)$ is a connected Lie subgroup of G and every connected Lie subgroup of G is contained in $\lambda(G)$). We call $\lambda(G)$ the *Lie radical* of G. Easy examples show that this property is not available in infinite-dimensional compact groups (even connected compact abelian groups may fail to posses such a subgroup $\lambda(G)$, take for example $G = \mathbb{T}^{\mathbb{N}}$)

Lemma 2.6. *Every compact finite-dimensional group G contains a largest connected Lie subgroup $\lambda(G)$. If $f : G \to H$ is a continuous homomorphism of compact finite-dimensional groups, then $f(\lambda(G)) \leq \lambda(H)$. In particular, $\lambda(G)$ is a fully invariant subgroup of G.*

Proof. Since every connected subgroup of G is contained in $c(G)$, it is clear that we can assume without loss of generality that G is also connected.

Consider first the case when G is abelian. Then a connected Lie subgroup of G is necessarily a torus, so we have to show that the family $\mathfrak{T}(G)$ of all tori in G has a largest torus. Note first that $\mathfrak{T}(G)$ is directed, as for $T_1, T_2 \in (G)$ the subgroup $T_1 + T_2$ is a torus (being isomorphic to a quotient of the torus $T_1 \times T_2$). Hence, it suffices to see that $\mathfrak{T}(G)$ contains a maximal torus (i.e., not properly contained in any other torus). It follows from Fact 2.2(a), that if $T_1, T_2 \in \mathfrak{T}(G)$ with $T_1 \leq T_2$, then either $T_1 = T_2$, or $\dim T_2 > \dim T_1$ (as T_2/T_1 is still a torus). Since G is finite-dimensional, there are exists a torus $T_{\max}$ of maximal dimension d. By the previous observation, $T_{\max}$ is maximal,.

In the general case, G' is a connected Lie group by Fact 2.4. On the other hand, G/G' is a connected finite-dimensional compact abelian group. So it has a largest torus $\lambda(G/G')$. Let $q : G \to G/G'$ be the canonical homomorphism. Then $B := q^{-1}(\lambda(G/G'))$ is a closed subgroup of G containing the connected subgroup G' and such that $B/G' \cong \lambda(G/G')$ is a compact connected Lie group. Then B is a connected Lie group as well. Assume that L is a connected Lie subgroup of G. Then $L \cdot G'$ is still a connected Lie subgroup of G. So, $q(L \cdot G') = q(L)$ is a connected Lie subgroup of G/G'. Hence, $q(L) \leq \lambda(G/G')$, and consequently, $L \leq q^{-1}(L \cdot G') \leq B$. Therefore, $B = \lambda(G)$ is the largest connected Lie subgroup of G. $\square$

3 Topological entropy of endomorphisms of compact groups

Let us briefly recall the definition of topological entropy following [1]. For a compact space X denote by $\mathrm{cov}(X)$ be the family of all open covers of X and for $\mathcal{U} \in \mathrm{cov}(X)$ put $H(\mathcal{U}) = \min\{\log |\mathcal{V}| : \mathcal{V}$ a finite subcover of $\mathcal{U}\}$. For a continuous self-map $\psi : X \to X$, $n \in \mathbb{N}_+$ and $\mathcal{U} \in \mathrm{cov}(X)$ let

$$C_n(\mathcal{U}, \psi) := \{U_0 \cap \psi^{-1}(U_1) \cap \ldots \cap \psi^{-n+1}(U_{n-1}) : U_k \in \mathcal{U}, k = 0, 1, \ldots, n-1\}.$$

The limit $H_{top}(\psi, \mathcal{U}) = \lim_{n\to\infty} \frac{H(C_n(\mathcal{U},\psi))}{n}$ exists by a folklore fact referred to as the Fekete Lemma. The *topological entropy* of ψ is

$$h_{top}(\psi) = \sup\{H_{top}(\psi, \mathcal{U}) : \mathcal{U} \in \mathrm{cov}(X)\}.$$

Now we recall the definition of Bowen's entropy in its more general form for uniform spaces from [18]. Let $(X,\mathcal{U})$ be a uniform space and let $\alpha : X \to X$ be a uniformly continuous map. For an entourage $U \in \mathcal{U}$, a subset F of X is said to (n,U)-*span* a compact subset C of X if for every $x \in C$ there is $y \in F$ such that $\big(\alpha^j(x), \alpha^j(y)\big) \in U$ for each $0 \leq j < n$. Let $r_n(U,C)$ be the smallest cardinality of a set F which (n,U)-spans C and let $h_B(\alpha, C) = \sup_{U\in\mathcal{U}}\{\limsup_{n\to\infty} \frac{1}{n}\log r_n(U,C)\}$. The *Bowen entropy* $h_B(\alpha)$ of α with respect to the uniform structure $\mathcal{U}$ is defined as

$$h_B(\alpha) = \sup\{h_B(\alpha, C) : C \subseteq X, \text{ compact}\}. \tag{4}$$

According to [14, Corollary 2.14], for a compact uniform space $(X,\mathcal{U})$ every continuous self-map $f : X \to X$ is uniformly continuous and $h_B(f) = h_{top}(f)$. This is why from now on we only use h to briefly denote the Bowen entropy, keeping in mind that in the compact case it simply coincides with the topological entropy h_{top}.

If $f : G \to G$ is an endomorphism of a topological group G and N is a closed normal subgroup of G, we say that N is f-*invariant* if $f(N) \leq N$. In such a case f induces an endomorphism of the quotient group G/N that we denote by f/N.

Next we recall some results from [2]:

Lemma 3.1. *(a) ([2, Lemma 8.4]) If every compact subset of a uniform space $(X,\mathcal{U})$ is finite, then $h(\alpha) = 0$ for each uniformly continuous self-map α of $(X,\mathcal{U})$. In particular, if X is a topological group without infinite compact subsets, then $X \in \mathfrak{E}_0$.*

(b) ([2, Corollary 8.1]) If $\alpha : X \to X$ and Y is an α-invariant subspace, then $h(\alpha\restriction_Y) \leq h(\alpha)$. In particular, if G is a topological group and $\alpha : G \to G$ is a continuous endomorphism, then $h(\alpha) \leq h(\widetilde{\alpha})$.

(c) ([2, Corollary 8.5]) Let G be a topological group with $\widetilde{G} \in \mathfrak{E}_{<\infty}$, then also $G \in \mathfrak{E}_{<\infty}$.

Proposition 3.2. *If $\alpha : G \to G$ is an endomorphism of a topological abelian group such that $\alpha(U_i) \subseteq U_i$ for a family $\{U_i : i \in I\}$ of neighbourhoods of 0 that form a local base at 0, then α is continuous and $h(\alpha) = 0$.*

Proof. Fix a compact set C in G, $i \in I$ and $n \in \mathbb{N}$. By the compactness of C there exists a finite set $F \subseteq G$ such that $C \subseteq \bigcup_{x\in F}(x + U_i)$. To see that C is (n,U_i)-spanned by F pick $y \in C$ and find $x \in F$ such that $y - x \in U_i$. Then $\alpha^j(y) - \alpha^j(x) = \alpha^j(y - x) \in U_i$ for every $j \in \mathbb{N}$. Thus $r_n(U_i,C) \leq |F|$ and $\lim_n \frac{\log r_n(U_i,C)}{n} = 0$. Therefore, $h(\alpha) = 0$. □

The next corollary directly follows from Proposition 3.2. Recall that a subgroup H of G is said to be *fully invariant*, if $\alpha(H) \subseteq H$ for every endomorphism of G.

Corollary 3.3. *Let G be a topological abelian group having linear topology τ. Then:*

(a) $h(m_k^G) = 0$ for every integer k;

(b) if τ has a local base formed by fully invariant subgroups of G then every endomorphism of G is τ-continuous and has zero entropy.

Proposition 3.2 and its corollaries allow us to give series of topological groups in $\mathfrak{E}_0$.

Proposition 3.4. *$\mathfrak{E}_0$ contains the class $\mathfrak{O}$ of all Orsatti groups.*

Proof. It is easy to see that a group $G \in \mathfrak{O}$ has as basic neighbourhoods of 0 all subgroups of G of the form nG, $n > 0$. Since the subgroups nG are obviously functorial, Corollary 3.3 yields $G \in \mathfrak{E}_0$. □

The topology used in the above proof is called $\mathbb{Z}$-*topology.* Orsatti [26] characterized $\mathfrak{O}$ as the class of abelian groups whose $\mathbb{Z}$-topology is compact.

Corollary 3.5. *Let (G, d) be a metric abelian group with invariant metric d. Then every non-expanding endomorphism of G is continuous and has zero entropy.*

Proof. Let α be a non-expanding endomorphism of G and let U_n denote the $1/n$-ball around 0 for every natural n. Then U_n form a base of neighbourhoods of 0 with $\alpha(U_n) \subseteq U_n$, hence Proposition 3.2 can be applied again. □

Corollary 3.6. *Let G be a zero-dimensional abelian group that is either locally compact or pseudocompact. Then $h(m_k^G) = 0$ for every integer k.*

Proof. Follows form Corollary 3.3 (a) as in both cases the zero-dimensional group G has a local base at 0 consisting of open subgroups (in the former case this follows from Fact 2.2 (b), for the latter case see [6]). □

The above corollary holds, in particular, for all compact abelian groups.

The following fact exhibits some fundamental properties of the topological entropy function in compact spaces.

Fact 3.7. *(a) (Logarithmic Law) Let X be a compact space and $\psi : X \to X$ a continuous self-map. Then $h(\psi^k) = kh(\psi)$ for every $k \in \mathbb{N}$; if $\psi : X \to X$ is a homeomorphism, then $h(\psi^{-1}) = h(\psi)$, so $h(\psi^k) = |k|h(\psi)$ for every $k \in \mathbb{Z}$.*

(b) (Reduction to surjective self-maps) Let X be a compact space and $\psi : X \to X$ be a continuous self-map. Then $E_\psi(X) := \bigcap_{n\in\mathbb{N}} \psi^n(X)$ is closed and ψ-invariant, the map $\psi\restriction_{E_\psi(X)}: E_\psi(X) \to E_\psi(X)$ is surjective and $h(\psi) = h(\psi\restriction_{E_\psi(X)})$.

(c) ("Continuity" w.r.t. inverse limits) Let (X_i, φ_{ij}, I) be an inverse system of compact spaces X_i with surjective connecting maps φ_{ij}, and canonical projections $\varphi_i : X = \varprojlim X_i \to X_i$. If $\psi_i : X_i \to X_i$, $i \in I$, are continuous maps with $\psi_i \circ \varphi_{i,j} = \varphi_{i,j} \circ \psi_j$ fors $j \leq i$ in I, then for the unique continuous map $\psi = \varprojlim \psi_i : X \to X$ with $\varphi_i \circ \psi = \psi_i \circ \varphi_i$ for every $i \in I$, one has $h(\psi) = \sup_{i\in I} h(\psi_i)$.

We shall often make use of the so called *Addition Theorem for entropy*:

Fact 3.8. [5, Theorem 19] *Let $f : G \to G$ be an endomorphism of a compact group G and let N be a closed normal f-invariant subgroup of G. Then*

$$h(f) = h(f\restriction_N) + h(f/N). \tag{5}$$

Remark 3.9. (a) This theorem was proved in [38] in the case of compact metrizable groups. The general cases can be obtained from this particular case, as shown in [2, Theorem 8.3].

(b) It is not known whether the Addition Theorem holds true for locally compact abelian groups (see [14]). Nevertheless, it remains true when G is arbitrary and N is open; as $h(f) = h(f\restriction_N)$ and $h(f/N) = 0$ in such a case (see [14, Corollary 4.17]). For zero-dimensional locally compact groups (5) was proved for some special cases in [17].

We explicitly formulate the next lemma for the sake of easy reference (items (a), (b) are trival) .

Fact 3.10. *Let k be an integer, let G be a topological abelian group and let H be a subgroup of G. Then:*

(a) H is m_k^G-invariant, $m_k^H = m_k^G \restriction_H$ and $m_k^G/H = m_k^{G/H}$;

(b) if G is compact and H is closed, then $h(m_k^G) = h(m_k^H) + h(m_k^{G/H})$;

(c) ([5, Theorem 15], [39]) $h(m_k^{\mathbb{R}^n}) = h(m_k^{\mathbb{T}^n}) = h(m_k^{\mathbb{K}^n}) = n \log k$ for every $k \in \mathbb{N}$.

It is easy to see that the class $\mathfrak{O}$ (see Definition 1.1) is closed under taking closed subgroup, quotients and extensions (so in particular, finite products). The larger classes of compact groups in $\mathfrak{E}_{<\infty}$ and $\mathfrak{E}_0$ are closed under taking topological direct summands:

Lemma 3.11. *Let G, H be compact groups. If $G \times H \in \mathfrak{E}_{<\infty}$, then $G, H \in \mathfrak{E}_{<\infty}$. The same holds for $\mathfrak{E}_0$.*

It easily follows from the Logarithmic Law 3.7 that if $G^\omega \in \mathfrak{E}_{<\omega}$, then $G \in \mathfrak{E}_0$.

We are not aware whether the classes $\mathfrak{E}_{<\infty}$ and $\mathfrak{E}_0$ are stable under taking finite products, and more generally, under taking extensions (see Question 7.3). However, one can see that this property is available when the extension is taken with respect to a closed fully invariant subgroup:

Proposition 3.12. *Let G be a compact group and let N be a closed fully invariant subgroup of G. If both $N, G/N \in \mathfrak{E}_{<\infty}$ (resp., $\mathfrak{E}_0$), then $G \in \mathfrak{E}_{<\infty}$ (resp., $G \in \mathfrak{E}_0$).*

Proof. If α is an endomorphism of G, then $c(G)$ is α-invariant and $h(\alpha \restriction_N) < \infty$ (resp., $h(\alpha \restriction_N) = 0$) by hypothesis. Since $G/N \in \mathfrak{E}_{<\infty}$ (resp., $\mathfrak{E}_0$), one has also $h(\alpha/N) < \infty$ (resp., $h(\alpha/N) = 0$). Now the Addition Theorem gives $h(\alpha) < \infty$ (resp., $h(\alpha) = 0$). □

This proposition will be applied in the following prominent cases:

(a) $N = c(G)$ is the connected component of G;

(b) $N = \overline{G'}$ is the closure of the commutator subgroup of G.

(c) $N = \lambda(G)$ is the largest connected Lie subgroup of G in case $\dim G < \infty$;

Let n be a positive integer, let $f(t) = st^n + a_{n-1}t^{n-1} + \ldots + a_0 \in \mathbb{Z}[t]$ be a non-constant polynomial with integer coefficients and let $\lambda_1, \ldots, \lambda_n$ be all complex roots of $f(t)$ taken with their multiplicity; so that $f(t) = s \cdot \prod_{i=1}^n (t - \lambda_i)$. The *(logarithmic) Mahler measure* of $f(t)$ was defined by Lehmer [22] (and independently by Mahler) by

$$m(f) = \log |s| + \sum_{|\lambda_i|>1} \log |\lambda_i|.$$

The Mahler measure of an endomorphism $\psi : \mathbb{Q}^n \to \mathbb{Q}^n$ is $m(\psi) = m(f)$, where $f(t)$ is the primitive polynomial with integer coefficients obtained from the characteristic polynomial of ψ after elimination of the denominators. More generally, if X is a rank n subgroup of $\mathbb{Q}^n$, then every endomorphism $\psi : X \to X$ admits a unique extension $\widetilde{\psi} : \mathbb{Q}^n \to \mathbb{Q}^n$. We let $m(\psi) := m(\widetilde{\psi})$. In case $X = \mathbb{Z}^n$, the polynomial $m(\psi)$ is monic. Finally, one can define the Mahler measure of a continuous endomorphism φ of $\mathbb{K}^n$ or $\mathbb{T}^n$ (or more generally, any finite-dimensional compact connected abelian group K), by letting $m(\varphi) := m(\widehat{\varphi})$ (note that $\widehat{K}$ is a finite-rank torsion-free abelian group and $\widehat{\varphi}$ is an endomorphism of $\widehat{K}$).

The next theorem combines together the Kolmogorov-Sinai Formula and the Yuzvinski Formula (see [39]) giving the value of the topological entropy of a continuous automorphism of $\mathbb{T}^n$ or $\mathbb{K}^n$ as its Mahler measure.

Theorem 3.13. *Let $n \in \mathbb{N}_+$ and let φ be a topological endomorphism of $\mathbb{T}^n$ or $\mathbb{K}^n$. Then $h(\varphi) = m(\varphi)$.*

Example 3.14. Theorem 3.13 implies that $\mathbb{T}^n \in \mathfrak{E}_\infty$ and $\mathbb{K}^n \in \mathfrak{E}_\infty$ for every $n \in \mathbb{N}$.

Using ideas briefly sketched in [1], Weiss [36] developed the definition of algebraic entropy for endomorphisms of abelian groups as follows. Let G be an abelian group and let $\phi : G \to G$ be an endomorphism of G. For a finite subset F of G and $n \in \mathbb{N}$, let $T_n(\phi, F) := F + \phi(F) + \ldots + \phi^{n-1}(F)$. The limit

$$H(\phi, F) := \lim_{n\to+\infty} \frac{\log |T_n(\phi, F)|}{n}, \tag{6}$$

exists by the Fekete Lemma (see [10], this definition was inspired by although different from that given by Peters [28]). The *algebraic entropy* of $\phi : G \to G$ is

$$h_{alg}(\phi) = \sup\{H(\phi, F) : F \text{ is a finite subset of } G\}. \tag{7}$$

The entropy defined in [1, 36] was making use of just finite *subgroups* of G

$$\mathrm{ent}(\phi) = \sup\{H(\phi, F) : F \text{ is a finite subgroup of } G\}. \tag{8}$$

Clearly both entropies coincide in case G is a torsion group.

Theorem 3.15 (Bridge Theorem)**.** [9] *Let G be an abelian group and $\phi : G \to G$ an endomorphism. Then*

$$h_{alg}(\phi) = h(\widehat{\phi}).$$

Algebraic entropy can be defined also in locally compact groups [29, 33].

4 Proof of Theorems A, B and C

4.1 Proof of Theorem A.

Before starting the proof of (2) from Theorem A, we note that the implication (b) $\to$ (c) is trivial. The implication (a) $\to$ (b) follows from Corollary 3.6 and the fact that the totally disconnected ω-bounded or locally compact groups are zero-dimensional (see [6] and [19], resp.). This will be used in the proof of (2). On the other hand, we can use the formula (2) to deduce the implication (c) $\to$ (a), since the conjunction of (c) and (2) entails $\dim G = 0$, hence total disconnectedness of G. The rest of the proof is dedicate to the verification of (2).

(A) Consider first the case when G is compact.

Assume $\dim G$ is infinite. By Lemma 2.2(d), there exists a continuous surjective homomorphism $f : G \to \mathbb{T}^\omega$. In particular, for every n there exists a continuous surjective homomorphism $f_n : G \to \mathbb{T}^n$. By Fact 3.10 (b) applied to G and its quotient $\mathbb{T}^n$, along with Fact 3.10 (c), $h(m_k^G) \geq n \log k$ for every n. Therefore, $h(m_k^G) = \infty$. This proves (2) when $\dim G$ is infinite.

Assume from now on that $d = \dim G < \infty$. By Lemma 2.2(d), there exists a continuous surjective homomorphism $f : G \to \mathbb{T}^d$. Since the discrete dual group $\widehat{G}$ has free rank d, the divisible hull D of $\widehat{G}$ is isomorphic to $\mathbb{Q}^d \times D_1$, where D_1 is a divisible torsion abelian group. Hence $N = \widehat{D_1}$ is a compact totally disconnected abelian group and $\widehat{D} \cong H := \mathbb{K}^d \times N$. Taking the adjoint homomorphism of the inclusion $\widehat{G} \hookrightarrow D$ we obtain a continuous surjective homomorphism $l : H \to G$. By Fact 3.10

$$h(m_k^H) \geq h(m_k^G) \geq h(m_k^{\mathbb{T}^d}) = d \cdot \log k.$$

So it suffices to prove that the formula (2) holds true also for the group H, i.e., $h(m_k^H) = d \cdot \log k$. Note that m_k^H is the Cartesian product of d copies of $m_k^{\mathbb{K}} : \mathbb{K} \to \mathbb{K}$ and the endomorphism $m_k^N : N \to N$. Since N is totally disconnected, $h(m_k^N) = 0$, by the implication (a) $\to$ (b) proved above. By the Addition Theorem and Fact 3.10 (c), we have $h(m_k^H) = h(m_k^{\mathbb{K}^d}) = d \cdot h(m_k^{\mathbb{K}}) = d \cdot \log k$.

(B) Now assume that G is ω-bounded and let K denote the compact completion of G. Then $\dim G = \dim K$, as G is pseudocompact [32]. Let $d = \dim G$ in case $\dim G$ is finite, and $d = \omega$ otherwise. According to Lemma 3.1 and case (A), $h(m_k^G) \leq h(m_k^K) = d \log k$. If we prove that there exists a compact subgroup N of G with $\dim N \geq d$, then $h(m_k^G) \geq h(m_k^N) \geq d \log k$ and we are done. Since $\dim K = d$, there exists a surjective continuous homomorphism $f : K \to \mathbb{T}^d$, by Lemma 2.2(d). The restriction of f to G is surjective as $\mathbb{T}^d$ is metrizable and $f(G)$ is a dense ω-bounded subgroup of $\mathbb{T}^d$, so $f(G) = \mathbb{T}^d$. Thus we get a surjective continuous homomorphism $f \restriction_G : G \to \mathbb{T}^d$. Let D be a dense cyclic subgroup of $\mathbb{T}^d$ and let D_1 be a cyclic subgroup of G such that $f(D_1) = D$. Then the closure N of D_1 in K is a compact subgroup of K contained in G and N has a quotient isomorphic to $\mathbb{T}^d$. Hence $\dim N \geq d$ and we are done.

(C) Now assume that G is locally compact. Then $G = \mathbb{R}^n \times G_0$, where G_0 contains an open compact subgroup K. Let $H = \mathbb{R}^n \times K$. Then H is an open subgroup of G, thus G/H is discrete. Hence the additivity theorem can be applied here, according to Remark 3.9, i.e., $h(m_k^G) = h(m_k^H)$. A further application of that theorem (possible since K is compact) gives

$$h(m_k^H) = n \log k + h(m_k^K) = n \log k + \dim K \log k = (n + \dim K) \log k = \dim G \log k.$$

To prove the last assertion assume that $\dim G = \infty$. Then $h(m_2^G) = \dim G \cdot \log 2 = \infty$, by (2). Hence, $G \notin \mathfrak{E}_\infty$. This concludes the proof of Theorem A. $\square$

Let us mention that (2) remains true for also for $k = 1$ as long as G is finite-dimensional, since $h(id_G) = 0$.

In the pseudocompact case we have the following:

Corollary 4.1. *Let G be a pseudocompact abelian group. Then $h(m_k^G) \leq \dim G \cdot \log k$ for every integer $k > 1$.*

Proof. Let K be the compact completion of G. Then $\dim K = \dim G$ ([32]), so $h(m_k^G) \leq h(m_k^K) \leq \dim K \cdot \log k$ by (2) and Lemma 3.1. $\square$

The corollary remains true, with the same proof, for locally pseudocompact groups which share similar properties with pseudocompact groups (see [32]).

We show in Example 4.2 that the hypothesis in Theorem A cannot be weakened to "countably compact" even for connected torsion-free abelian groups.

Example 4.2. In [15] a forcing model of ZFC was built, such that every torsion-free abelian group G with $\mathfrak{c} \leq |G| \leq 2^{\mathfrak{c}}$ admits a connected countably compact group topology having no infinite compact subsets. Then, $G \in \mathfrak{E}_0$ by Lemma 3.1(a). Moreover, since G is not metrizable, we deduce that $\widetilde{G}$ is not metrizable either. As connected finite-dimensional compact groups are metrizable, we deduce that $\dim G = \dim \widetilde{G} = \infty$. So $h(m_k^G) = 0$ (while $h(\widetilde{m_k^G}) = \infty$) for every $k > 1$ and the equality (2) fails for G.

4.2 Proof of Theorems B and C.

For the proof of Theorem B we need to check that compact semi-simple Lie groups belong to $\mathfrak{E}_0$.

Remark 4.3. Let L be a compact semi-simple Lie group. We prove that $L \in \mathfrak{E}_0$.

(a) Let us first check that $L \in \mathfrak{E}_0$ under the stronger assumption that L is simple and center-free. It is known that for an automorphism ψ of a simple Lie group L some finite power ψ^n must be an inner automorphism ([37]), and consequently, have zero entropy (see Example 1b in [1]). Hence, also ψ has zero entropy by the Logarithmic Law 3.7. This proves that $L \in \mathfrak{E}_0$.

(b) Now we prove that $L \in \mathfrak{E}_0$ in the general case. Every endomorphism α of L is continuous by van der Waerden's theorem [34]. According to Lemma 3.1(b), $E_\alpha(L) = \bigcap_m \alpha^m(L)$ is α-invariant, $\alpha' := \alpha \restriction_{E_\alpha(L)}: E_\alpha(L) \to E_\alpha(L)$ is surjective and $h(\alpha) = h(\alpha \restriction_{E_\alpha(L)})$. The decreasing chain $L \supseteq \alpha(L) \supseteq \alpha^2(L) \supseteq \ldots$ stabilises, as each member $\alpha^k(L)$ is a connected Lie group, so each proper inclusion corresponds to a proper decrease of dimension. In particular, $E_\alpha(L) = \bigcap_m \alpha^m(L)$ coincides with some $\alpha^m(L)$, so it is a connected compact semi-simple Lie group in its own turn. This shows that in order to check $h(\alpha) = 0$, we may simply assume that α is surjective from the very beginning. Then the center $Z(L)$ is α-invariant, hence the induced endomorphism $\bar{\alpha} := \alpha/Z(L)$ of $L^* = L/Z(L)$ is surjective. Since $Z(L)$ is finite, $h(\alpha \restriction_{Z(L)}) = 0$. Hence, (5) implies that $h(\alpha) = h(\bar{\alpha})$. Since L^* is center-free, $L^* = \prod_{n=1}^m S_n$, where each S_n is a simple center-free Lie group. Therefore, the only closed normal subgroups of L^* are the subproducts of the form $\prod_{i=1}^t S_{n_i}$ for appropriate indexes $1 \leq n_1 < n_2 < \ldots < n_t \leq m$. Since every non-trivial product of this form has positive dimension and $\bar{\alpha}$ is surjective, we deduce, with Fact 2.2(a), that $\ker \bar{\alpha}$ is trivial. Hence, $\bar{\alpha}$ is an automorphism. Then $\bar{\alpha}$ acts as a bijection on the set of coordinate groups S_n, so an appropriate power $\bar{\alpha}^s$ of $\bar{\alpha}$ is a direct product of automorphisms of the single components S_n. Since each of these automorphisms has entropy zero by item (a), we conclude that $h(\bar{\alpha}^s) = 0$, by (5). From the Logarithmic Law we get $h(\bar{\alpha}) = 0$, and finally $h(\alpha) = 0$. Thus, $L \in \mathfrak{E}_0$.

Another ingredient of the proof of Theorem B is the following:

Proposition 4.4. *Let G be a connected pseudocompact group. If G is finite-dimensional, then G is compact and $G \in \mathfrak{E}_{<\infty}$.*

Proof. Assume that G is finite-dimensional. Then its completion $\widetilde{G}$ is connected and $\dim G = \dim \widetilde{G} = n < \infty$. Since finite-dimensional compact connected groups are metrizable, we deduce that $G = \widetilde{G}$ is a compact metrizable group.

Consider first the case when G is abelian. Then there exists a continuous surjective homomorphism $l : \mathbb{K}^n \to G$. Indeed, since $\dim G = n$, the free rank of the Pontryagin dual $X = \widehat{G}$ is n, hence X is isomorphic to a subgroup of the group $\mathbb{Q}^n$. Assume without loss of generality that it is actually a subgroup of $\mathbb{Q}^n$ and let $i : X \hookrightarrow \mathbb{Q}^n$ be the inclusion homomorphism. Taking the adjoint of i we obtain the desired l. Let $\alpha : G \to G$ be a continuous endomorphism of G. Its adjoint is a group homomorphism $\widehat{\alpha} : X \to X$. Since X is a subgroup of $\mathbb{Q}^n$ and the latter group is divisible, there exists an extension $s : \mathbb{Q}^n \to \mathbb{Q}^n$ of $\widehat{\alpha}$. Now consider $\widehat{s} : \mathbb{K}^n \to \mathbb{K}^n$. Since $s \cdot i = i \cdot \widehat{\alpha}$, it follows that $l \cdot \widehat{s} = \alpha \cdot l$. By (5), applied to $\mathbb{K}^n, \widehat{s}$ and $\ker \widehat{s}$, we deduce that $h(\alpha) \leq h(\widehat{s})$. Now it remains to recall that $h(\widehat{s}) < \infty$ as $\mathbb{K}^n \in \mathfrak{E}_{<\infty}$, according to Example 3.14.

Consider now the general case. Denote by $Z(G)$ the center of G and let $A = c(Z(G))$. Then $G = A \cdot G'$. Since $\dim G' < \infty$, we deduce that G' is a semi-simple Lie group (Fact 2.4). According to Remark 4.3 (b), $G' \in \mathfrak{E}_0$. Since G' is closed, Proposition 3.12(b) implies that $G \in \mathfrak{E}_{<\infty}$ iff $G/G' \in \mathfrak{E}_{<\infty}$.

As G/G' is a compact connected abelian group with $\dim G/G' \leq \dim G < \infty$, we have $G/G' \in \mathfrak{E}_{<\infty}$ by the above argument. □

According to Proposition 4.4, every continuous endomorphism of a finite-dimensional compact connected abelian group G has finite entropy. Example 6.8 shows the converse need not be true in the non-abelian case.

Now we show that for connected abelian groups G close to being compact, $G \in \mathfrak{E}_{<\infty}$ if and only if $\dim G < \infty$.

Corollary 4.5. *A connected abelian group G that is either ω-bounded or locally compact admits a continuous endomorphism of infinite entropy if and only if G is infinite-dimensional.*

Proof. Follows from Corollary 4.1 and Proposition 4.4. □

Example 4.2 shows that the hypothesis "ω-bounded" in the above corollary cannot be weakened to "countably compact".

Proof of Theorem B. We have to prove that for a compact abelian group G:

(a) if $\dim G < \infty$ and $G/c(G) \in \mathfrak{E}_{<\infty}$, then also $G \in \mathfrak{E}_{<\infty}$.

(b) $G \in \mathfrak{E}_0$ iff G is totally disconnected and $G = \prod_p G_p$, where $G_p \in \mathfrak{E}_0$ is a pro-p-group for every prime p.

(a) Assume that $\dim G < \infty$ and $G/c(G) \in \mathfrak{E}_{<\infty}$. By Proposition 4.4, $c(G) \in \mathfrak{E}_{<\infty}$, as $\dim c(G) < \infty$. Since $c(G)$ is a fully invariant subgroup of G and $G/c(G) \in \mathfrak{E}_{<\infty}$, we can apply Proposition 3.12 to conclude that $G \in \mathfrak{E}_{<\infty}$.

(b) Assume that $G \in \mathfrak{E}_0$, then $\dim G = 0$, by Theorem A. Therefore, $G = \prod_{p \in \mathbb{P}} G_p$ where each G_p is a pro-p-group. By Lemma 3.11, $G_p \in \mathfrak{E}_0$

Vice versa, assume that G is totally disconnected of the above form and $G_p \in \mathfrak{E}_0$ for every p. Pick a continuous endomorphism $f : G \to G$. Then each G_p is fully invariant, hence $f(G_p) \leq G_p$. Let $f_p = f\restriction_{G_p}$. By our hypothesis, $h(f_p) = 0$ for all p. By the Addition Theorem, applied the restriction $f_{(n)}$ of f to the product P_n of the first n members of $G = \prod_{p \in \mathbb{P}} G_p$, $h(f_{(n)}) = 0$. By the continuity of entropy with respect to inverse limits (Fact 3.7(c)), $h(f) = \lim_n h(f_{(n)}) = 0$. Therefore, $G \in \mathfrak{E}_0$. □

Lemma 4.6. *Suppose that for a connected group G the subgroup $Z(G')$ has finite exponent.*

(a) If $G \in \mathfrak{E}_{<\infty}$, then $\dim Z(G) < \infty$;

(b) If $G \in \mathfrak{E}_0$, then $G = G'$.

Proof. Let $A := c(Z(G))$ and L be as in Fact 2.4 (a). Let $\eta : A \times L \to G$ the continuous surjective homomorphism with kernel $N \leq A \times Z(L)$, as in Fact 2.4 (b). By hypothesis, $\exp(Z(L)) < \infty$. Let $k = 2\exp(Z(L))$ and consider the endomorphism $f = (m_A^k \times o_L)$ of $A \times L$, where $\ker o_L = L$. Since its restriction to $A \times Z(L)$ coincides with $m^k_{A \times Z(L)}$ and the latter endomorphism of $A \times Z(L)$ leaves invariant all subgroups of $A \times Z(L)$, we have $f(N) \leq N$. Therefore, f induces an endomorphism $\overline{f} : G \to G$. Since the restriction of $\overline{f}$ to $\eta(A)$ coincides with $m^k_{\eta(A)}$, one has:

$$\dim A \cdot \log k = h(m_A^k) = h(m^k_{\eta(A)}) \leq h(\bar{f}). \tag{9}$$

(a) Suppose that $G \in \mathfrak{E}_{<\infty}$. Then $h(\overline{f}) < \infty$, so (9) yields $\dim A \cdot \log k < \infty$. Since $k > 1$, this yields $\dim A < \infty$.

(b) Assume now that $G \in \mathfrak{E}_0$. Similarly, from $h(\overline{f}) = 0$ and (9) we deduce that $\dim A \cdot \log k = 0$ and $\dim A = 0$, as $k > 1$. This yields $A = \{0\}$ by Theorem B(b), hence $G = G'$. □

Proposition 4.7. *If L is a compact connected Lie group, then $G \in \mathfrak{E}_{<\infty}$ for every closed subgroup of L (i.e., $L \in \mathsf{S}(\mathfrak{E}_{<\infty})$). Moreover, $G \in \mathfrak{E}_0$ if and only if $Z(G)$ is finite.*

Proof. As a closed subgroup of a Lie group, the group G is a Lie group itself. Then $c(G)$ is an open fully invariant finite-index subgroup of G, so $G/c(G) \in \mathfrak{E}_0$, being finite. According to Proposition 3.12, it suffices to show that $c(G) \in \mathfrak{E}_{<\infty}$ (or $c(G) \in \mathfrak{E}_0$, in case $Z(G)$ is finite). In other words, we can replace G by the group L itself and prove that $L \in \mathfrak{E}_{<\infty}$ (resp., $L \in \mathfrak{E}_0$, in case $Z(L)$ is finite).

Since $c(L)'$ is a semi-simple Lie group, $c(L)' \in \mathfrak{E}_0$, by Remark 4.3 (b). As $c(L) = c(L)' \cdot Z(c(L))$, it suffices to note that $c(L)/c(L)' \cong Z(c(L))/(Z(c(L)) \cap c(L)') \in \mathfrak{E}_{<\infty}$, being a finite-dimensional connected abelian Lie group (i.e., a torus), by Example 3.14. Now Proposition 3.12 applies again.

If $Z(L)$ is finite, then $L \in \mathfrak{E}_0$, by Proposition 3.12. Hence, $L = L'$ by Lemma 4.6. Since L is a Lie group, $L = L'$ means that L is semi-simple, i.e., $Z(L)$ is finite. □

Proof of Theorem C. We have to prove that for a compact connected group G with $\dim G' < \infty$:

(a) $G \in \mathfrak{E}_{<\infty}$ if and only if $\dim G < \infty$;

(b) $G \in \mathfrak{E}_0$ if and only if $G = G'$ (if and only if $\dim Z(G) = 0$ if and only if $Z(G)$ is finite).

By $\dim G' < \infty$ and Remark 2.5, we deduce that G' is a connected semi-simple Lie group. In particular, $G' \in \mathfrak{E}_0$, by Proposition 4.7.

(a) Let $A := c(Z(G))$. As $\dim A \cap G' = 0$, we deduce, with Fact 2.2, that $\dim G/A \cap G' = \dim G$ and $\dim A/A \cap G' = \dim A$. On the other hand, the equality $G = A \cdot G'$ yields $G/(A \cap G') \cong A/(A \cap G') \times G'/(A \cap G')$. Hence,

$$\dim G = \dim A/(A \cap G') + \dim G'/(A \cap G') = \dim A + \dim G'.$$

Therefore, the hypothesis $\dim G' < \infty$ allows us to claim that $\dim G < \infty$ precisely when $\dim A = \dim Z(G) < \infty$.

In view of $G' \in \mathfrak{E}_0$, $G \in \mathfrak{E}_{<\infty}$ if $G/G' \cong A/(A \cap G') \in \mathfrak{E}_{<\infty}$, according to Proposition 3.12. By Theorem B, this is equivalent to $\dim A/(A \cap G') = \dim A < \infty$. This proves the implication "$\Leftarrow$" in (a). To prove the remaining implication assume that $G \in \mathfrak{E}_{<\infty}$. As noted above, G' is a connected semi-simple Lie group, hence $Z(G')$ is finite. Hence, we can apply Lemma 4.6 to deduce that $\dim Z(G) < \infty$. As noted above, this yields $\dim G < \infty$. This concludes the proof of item (a).

(b) As $G' \in \mathfrak{E}_0$ by Remark 4.3(b), $G \in \mathfrak{E}_0$ if $G/G' \cong A/A \cap G' \in \mathfrak{E}_0$, according to Proposition 3.12. By Theorem B, this is equivalent to $G = G'$.

Now suppose that $G \in \mathfrak{E}_0$. As mentioned above, our hypothesis about G' allows us to apply Lemma 4.6 to deduce that $G = G'$.

Finally, note that $G = G'$ yields $|Z(G)| < \infty$ (so, $\dim Z(G) = 0$), as G' is a Lie group. On the other hand, $\dim Z(G) = 0$ yields $G = G'$, by Fact 2.4. □

5 Compact abelian groups with endomorphisms of infinite entropy

This section leads us out of the class $\mathfrak{E}_{<\infty}$ of groups without endomorphisms of infinite entropy. It was proved in [2, Theorem 8.4] that for every infinite *discrete* abelian group G the Bohr compactification bG has this property, i.e., $bG \notin \mathfrak{E}_{<\infty}$. We provide some simpler examples below:

Example 5.1. Let K be a compact group. The (*left*) *Bernoulli shift* $\beta_K : K^{\mathbb{N}} \to K^{\mathbb{N}}$ is defined by

$$\beta_K(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots).$$

It's entropy can be computed by the formula $h(\beta) = \log |K|$, where $\log |K|$ for an infinite K is intended as ∞. This fact, beyond Lemma 3.10, yields that $K^\kappa \in \mathfrak{E}_{<\infty}$ for a compact group K implies either $\kappa < \omega$ or $|K| < \infty$.

Example 5.2. Let p be a prime.

(a) $\mathbb{Z}(p^k)^\kappa \notin \mathfrak{E}_{<\infty}$ for every integer $k > 0$ and every infinite cardinal κ;

(b) $G = \prod_n \mathbb{Z}(p^{k_n}) \notin \mathfrak{E}_{<\infty}$ for every sequence k_n of natural numbers.

Proof. (a) Since κ is infinite, the group $H = \mathbb{Z}(p^k)^\kappa$ satisfies $H \cong H^{\mathbb{N}}$. Therefore, $H \notin \mathfrak{E}_{<\infty}$ as H is infinite, according to Example 5.1.

(b) Consider first the case when k_n is a strictly increasing sequence. Define $\alpha : G \to G$ as follows:

$$\alpha((x_1, x_2, \ldots, x_n, x_{n+1}, \ldots)) = (p^{k_2 - k_1} x_2, p^{k_3 - k_2} x_3, \ldots, p^{k_{n+1} - k_n} x_{n+1}, \ldots).$$

For arbitrarily $n \in \mathbb{N}$ and let $p_n : G \to \mathbb{Z}(p^{k_n})^{\mathbb{N}}$ be the surjective homomorphism defined by

$$p_n(x_1, x_2, \ldots) = (x_n, p^{k_{n+1} - k_n} x_{n+1}, \ldots, p^{k_m - k_n} x_m, \ldots) \in \mathbb{Z}(p^{k_n})^{\mathbb{N}}$$

for any $(x_1, x_2, \ldots) \in G$. As p_n is continuous and $p_n \circ \alpha = \beta_{\mathbb{Z}(p^{k_n})} \circ p_n$, we conclude that $h(\alpha) \geq h(\beta_{\mathbb{Z}(p^{k_n})}) = k_n \log p$, by the monotonicity of h. Hence, $h(\alpha) = \infty = \sup_n k_n$.

In the general case we apply Lemma 3.11, so it suffices to note that G has a direct summand that is either of the form $\mathbb{Z}(p^k)^\omega$ or $\prod_p \mathbb{Z}(p^{k_n})$ where $k \in \mathbb{N}$ and k_n is a strictly increasing sequence. Now (a) and the above argument apply. $\square$

As we saw above, for a group G as in item (b), $\{0, \infty\} \subseteq \mathbf{E}_{top}(G)$. Yet the argument gives no clue as to whether one has an equality here. We prove in Theorem D, that $\mathbf{E}_{top}(G) = \{0, \infty\}$ precisely when the sequence (k_n) is finitely many-to-one.

5.1 Proof of Theorems D and E

Proof of Theorem D: We have to prove the equivalence of the following conditions for an infinite totally disconnected compact abelian group K:

(a) $\mathbf{E}_{top}(K) \not\subseteq \{0, \infty\}$, i.e., $0 < h(f) < \infty$ for some continuous endomorphism $f : K \to K$;

(b) some of the Ulm-Kaplanski invariants $f_{n,p}(\widehat{K})$ of its Pontryagin dual $\widehat{K}$ are infinite, i.e., K is not semi-standard;

(c) K has a direct summand of the form $\mathbb{Z}(p^n)^{\mathbb{N}}$ for some $n \in \mathbb{N}$ and some prime p;

(d) $\mathbf{E}_{top}(K) \supseteq \{\infty\} \cup \{mn \log p : m \in \mathbb{N}\}$ for some prime p and $n \in \mathbb{N}$;

The implication (d) $\to$ (a) is trivial, while the implication (c) $\to$ (d) follows from Example 5.1, by making use of the powers β_K^m, for $K = \mathbb{Z}(p^n)^{\mathbb{N}}$.

The implication (b) $\to$ (c) easily follows from Fact **??**. Indeed, by Fact **??**, $\mathbb{Z}(p^n)^{(f_n)}$ is a direct summand of $\widehat{K}$. Thus, by standard properties of Pontryagin duality, $\mathbb{Z}(p^n)^{f_n}$ is a topological direct summand of K. Since f_n is infinite, we deduce that K has a direct summand of the form $\mathbb{Z}(p^n)^{\mathbb{N}}$ for some $n \in \mathbb{N}$ and some prime p.

The final part of the proof is dedicated to the proof of the remaining implication (a) $\to$ (b).

By our hypothesis, there exists a continuous endomorphism $f : K \to K$ with $0 < h(f) < \infty$. First we assume that that K is a pro-p-group with $\mathbf{E}_{top}(K) \neq \{0, \infty\}$. Then $G = \widehat{K}$ is a p-group. For the adjoint endomorphism $\phi = \widehat{f} : G \to G$ the algebraic entropy $\mathrm{ent}(\phi)$, as defined in §2, satisfies $\mathrm{ent}(\phi) = h(f)$, by Theorem 3.15. So, $0 < \mathrm{ent}(\phi) < \infty$. We adopt in the sequel the idea and the line of the proof of [11, Theorem 1.16], although we redesign the proof to make it accessible.

The hypothesis $\mathrm{ent}(\phi) > 0$ entails $\mathrm{ent}(\phi \restriction_{X[p]}) > 0$ by [11]. So there exists a non-zero element $x \in G[p]$ such that the orbit $T(\langle x \rangle, \phi) := \bigcup_n T_n(\phi, \langle x \rangle)$ is infinite.

Let $S_k = p^k G \cap T(\langle x \rangle, \phi)$ for every $k \in \omega$. Then

$$T(\langle x \rangle, \phi) = S_0 \supseteq S_1 \supseteq \ldots \supseteq S_n \supseteq \ldots.$$

First we prove that $S_m = 0$ for some m. Indeed, assume that $S_m \neq 0$ for some m and pick $0 \neq z \in S_m$. There exists $y \in G$ with $z = p^m y$, so $o(y) = p^{m+1}$. Let H be the subgroup of G generated by $\{\phi^k(y) : 0 \leq k < \infty\}$. Since $T(\langle x \rangle, \phi) = \bigoplus_{k=0}^{\infty} \langle \phi^k(x) \rangle$, one has $H = \bigoplus_{k=0}^{\infty} \langle \phi^k(y) \rangle$. Moreover, since $\phi \restriction_{T(\langle x \rangle, \phi)}$ acts as a Bernoulli shift, $\phi \restriction_H$ acts as a Bernoulli shift on the subgroup H. Then $\mathrm{ent}(\phi \restriction_H) = (m+1) \log p$. Therefore, the assumption $S_m \neq 0$ yields $\mathrm{ent}(\phi) > m \log p$. Since $\mathrm{ent}(\phi) < \infty$ we conclude that $S_m = 0$ for some m.

For $0 \leq k < m$ split $S_k = S_{k+1} \oplus S'_k$ for an appropriate subgroup S'_k of S_k. This gives

$$S_0 = S'_0 \oplus S'_1 \oplus S'_2 \oplus \ldots \oplus S'_{m-2} \oplus S_{m-1}.$$

Since S_0 is infinite, either S_{m-1} is infinite, or there exists $k < m-1$ such that S'_k is infinite. In the latter case, from $S'_k \cap S_{k+1} = \{0\}$, $S_{k+1} \subseteq p^{k+1} G[p]$ and $S'_k \subseteq p^k G[p]$, we conclude that $S'_k \hookrightarrow p^k G[p]/p^{k+1} G[p]$. Indeed,

$$S'_k \cap p^{k+1} G[p] = S'_k \cap T(\langle x \rangle, \phi) \cap p^{k+1} G[p] = S'_k \cap S_{k+1} = \{0\}.$$

Hence, the canonical homomorphism $q : p^k G[p] \to p^k G[p]/p^{k+1} G[p]$ when restricted to S'_k induces the desired monomorphism $S'_k \hookrightarrow p^k G[p]/p^{k+1} G[p]$. Therefore, we conclude that the Ulm-Kaplanski invariant $f_k := f_k(G)$ is infinite. In case S_{m-1} is infinite, we use the fact that $S_m = \{0\}$ and we deduce, as before, that $S_{m-1} \hookrightarrow p^{m-1} G[p]/p^m G[p]$ and conclude that the Ulm-Kaplanski invariant f_m is infinite.

In the general case the totally disconnected compact group K can be written in the form $K = \prod_p K_p$, where each K_p is a pro-p-group, by Lemma 2.1. If $\mathbf{E}_{top}(K_p) \subseteq \{0, \infty\}$ for all p, then also $\mathbf{E}_{top}(K) \subseteq \{0, \infty\}$ (by the Addition Theorem and Fact 3.7), contrary to our hypothesis. Hence, there exists a prime p with $\mathbf{E}_{top}(K_p) \nsubseteq \{0, \infty\}$. Then the above argument implies (b). $\square$

Proof of Theorem E. To prove (a) we have to show that if $G \in \mathfrak{E}_{<\infty}$ is totally disconnected, compact and abelian, then $w(G) \leq \mathfrak{c}$. By Lemma 2.2, G is a direct product of its pro-p-components. So, we can assume without loss of generality that G is a pro-p-group. Then our hypothesis $G \in \mathfrak{E}_{<\infty}$ and Theorem

D imply that the Pontryagin dual X of G is semi-standard. Write $X = X_1 \oplus D$, where $D = \mathbb{Z}(p^\infty)^{(\lambda)}$ is divisible and X_1 is reduced (i.e., contains no non-trivial divisible subgroups). Then $G \cong \widehat{X_1} \times \mathbb{Z}_p^\lambda$, Since $G \in \mathfrak{E}_{<\infty}$, we conclude that $\lambda < \infty$. Hence it suffices to prove that $|X_1| = w(\widehat{X_1}) = w(G_1) \leq \mathfrak{c}$. Since the reduced p-group, X_1 is semi-standard, we conclude that $|X_1| \leq \mathfrak{c}$, by [16].

(b) By Theorem 5.4 of [11] there exists a family of $2^{\mathfrak{c}}$ many pairwise non-isomorphic semi-standard p-groups $\{A_i : i \in I\}$ such that for each $i \in I$

(i) A_i length ω, i.e., $\bigcap_{n=1}^\infty p^n A_i = \{0\}$;

(ii) $\mathrm{ent}(\phi) = 0$ for all endomorphisms ϕ of A_i.

According to the Bridge Theorem 3.15, the property (ii) implies that $G_i = \widehat{A_i} \in \mathfrak{E}_0$ for every $i \in I$. To check that for all $i \in I$ $t(G_i)$ is dense in G_i we have to see that the annihilator $t(G_i)^\perp$ of $t(G_i)$ in A_i is trivial. This is clear since $t(G_i) = \bigcup_{n=1}^\infty G_i[p^n]$ and $G_i[p^n]^\perp = p^n A_i$, so $t(G_i)^\perp = \bigcap_{n=1}^\infty p^n A_i = \{0\}$. □

Remark 5.3. We proved in Theorem E that a totally disconnected $G \in \mathfrak{E}_{<\infty}$ is semi-standard and this leads to $w(G) \leq \mathfrak{c}$. Let us note that semi-standard totally disconnected compact groups need not necessarily belong to $\mathfrak{E}_{<\infty}$. Indeed, this is easy to see when the group is just a product of cyclic p-groups (see Example 5.2). Actually, the group $H = \prod_n \mathbb{Z}(p^n)$ has e-spectrum $\{0, \infty\}$.

6 Proof of Theorem F

For an isomorphism closed class $\mathfrak{P}$ of topological groups let $\mathsf{Q}(\mathfrak{P})$ ($S(\mathfrak{P})$, resp.) denote the class of all groups G such that all quotients (closed subgroups, resp.) of G belong to $\mathfrak{P}$. Clearly, $\mathsf{Q}(\mathfrak{P})$ ($\mathsf{S}(\mathfrak{P})$, resp.) is the largest subclass of $\mathfrak{P}$ stable under taking quotients (subgroups, resp.). In these terms, we characterize here the compact groups in $\mathsf{Q}(\mathfrak{E}_{<\infty})$ and $S(\mathfrak{E}_{<\infty})$ in two cases: abelian groups (in §6.1) and connected groups (in §6.2). This explains the blanket condition $\dim G < \infty$ (necessary for $G \in \mathsf{S}(\mathfrak{E}_{<\infty})$ according to Theorem A) implicitly present in §6.1. In §6.2, where commutativity is "traded" for connectedness, this condition plays a different role (see Theorems 6.7 and 6.9).

6.1 Abelian case: the class $\mathfrak{O}_c$ and its characterization via $\mathfrak{O}$, $\mathfrak{E}_{<\infty}$ and $\mathfrak{E}_0$

Definition 6.1. We denote by $\mathfrak{O}_c$ the class of all compact finite-dimensional abelian group G such that $G/c(G) \in \mathfrak{O}$.

In other words, this class is obtained by extending connected compact finite-dimensional abelian groups by Orsatti groups. We shall see below (Theorem 6.4) that they can also be obtained in the other way round: namely as extensions of Orsatti groups by means of connected compact finite-dimensional abelian groups. Indeed, if $N \in \mathfrak{O}$ is a closed subgroup of a compact abelian group G such that G/N is a connected finite-dimensional group, then the canonical homomorphism $G \to G/N$ takes $c(G)$ onto G/N, hence $G = c(G) + N$. Therefore, $G/c(G) \cong N/(N \cap c(G)$ so $G \in \mathfrak{O}_c$ as $\mathfrak{O}$ is stable under taking quotients.

The groups of the class $\mathfrak{O}_c$ are metrizable, since the finite-dimensional compact connected groups, as well as the groups in $\mathfrak{O}$, are all metrizable.

Proposition 6.2. *For every compact finite-dimensional abelian group G the following statements are equivalent:*

(a) $G \in \mathfrak{O}_c$.

(b) G/pG *is finite for every prime* p;

(c) there exists a closed subgroup G_1 *of* G *containing* $c(G)$*, such that* $G = G_1 \times \prod_p \mathbb{Z}_p^{n_p}$ *and* $G_1/c(G) \cong \prod_p F_p$*, where* n_p *is a non-negative integer and* F_p *is a finite* p*-group for every* $p \in \mathbb{P}$.

Proof. The implication (c) $\Rightarrow$ (b) trivially follows from the definition of $G_1, \mathfrak{O}$, the isomorphisms

$$G/c(G) \cong (G_1/c(G)) \times \prod_p \mathbb{Z}_p^{n_p} \cong \left(\prod_p F_p\right) \times \left(\prod_p \mathbb{Z}_p^{n_p}\right) \cong \prod_p \left(\mathbb{Z}_p^{n_p} \times F_p\right).$$

and the fact that G/pG is a quotient of $G/c(G)$. To check the implication (b) $\Rightarrow$ (a) assume that $|G/pG| < \infty$ for every prime p. Since $c(G) \leq pG$ for every p, the group $H = G/c(G)$ satisfies $H/pH < \infty$ for every prime p as well. According to Lemma 2.1, $H = \prod_p H_p$ as a product of pro-p-groups. Then $H/pH \cong H_p/pH_p$ for all $p \in \mathbb{P}$. For $X_p = \widehat{H_p}$ one has $X[p] \cong H_p/pH_p$ [12], so $r_p(X_p) < \infty$. Hence $X_p \cong \mathbb{Z}(p^\infty)^{s_p} \times F_p$ for some $s_p \in \mathbb{N}$ and some finite p-group F_p. Then $H_p \cong \mathbb{Z}_p^{s_p} \times F_p$, so $G/c(G) = H \cong \prod_p (\mathbb{Z}_p^{s_p} \times F_p) \in \mathfrak{O}$. This proves $G/c(G) \in \mathfrak{O}$, so $G \in \mathfrak{O}_c$.

We are left now with the proof of the missing implication (a) $\Rightarrow$ (c). The hypothesis $G/c(G) \in \mathfrak{O}$ yields $G/c(G) \cong \prod_p (\mathbb{Z}_p^{s_p} \times F_p)$, with $n_p \in \mathbb{N}$ and finite p-groups F_p for every prime p. Let $N = \prod_p F_p$ and consider the canonical homomorphism $f : G \to G/c(G)$. Set $G_1 = f^{-1}(N)$, so that $G_1/c(G) \cong \prod_p F_p$ and $G/G_1 \cong \prod_p \mathbb{Z}_p^{n_p}$. According to Lemma 2.1, G_1 splits topologically in G, i.e., $G = G_1 \times \prod_p \mathbb{Z}_p^{n_p}$. $\square$

Lemma 6.3. *The classes* $\mathfrak{O}$ *and* $\mathfrak{O}_c$ *are stable under taking closed subgroup, quotients and extensions.*

Proof. The stability of $\mathfrak{O}$ under taking closed subgroup, quotients and extensions is clear.

The stability of $\mathfrak{O}_c$ under taking quotients easily follows from item (b) of the above proposition and the fact that if $f : G \to G_1$ is a continuous surjective homomorphism of compact groups, then G_1/pG_1 is isomorphic to a quotient of G/pG. The stability of $\mathfrak{O}_c$ under extension is equally easy to check.

To prove that $\mathfrak{O}_c$ is stable under taking closed subgroups take $G \in \mathfrak{O}_c$ and a closed subgroup N of G.

Case 1. G is connected. Let $n = \dim G$. Then there exists a continuous surjective homomorphism $f : \mathbb{K}^n \to G$ such that $H = \ker f$ is totally disconnected [12].

Subcase 1.1. Assume first that N is totally disconnected. Since H is totally disconnected, $L = f^{-1}(N)$ is a closed totally disconnected subgroup of $\mathbb{K}^n$ containing H. According to [12], there exists a closed subgroup $\mathbb{H} \cong \prod_p \mathbb{Z}_p^n$ of $\mathbb{K}^n$ such that $\mathbb{K}^n/\mathbb{H} \cong \mathbb{T}^n$. Let $q : \mathbb{K}^n \to \mathbb{K}^n/\mathbb{H} \cong \mathbb{T}^n$ be the canonical homomorphism. Then $q(L)$ is a closed totally disconnected subgroup of $\mathbb{T}^n$. As $\mathbb{T}^n$ is a Lie group, this yields that $q(L)$ is finite. Let $m = |q(L)|$. Then $mq(L) = 0$, so $mL \subseteq \mathbb{H}$, thus $L \cong mL = \prod_p \mathbb{Z}_p^{k_p}$ for appropriate $k_p \leq n$ for each p (see Lemma 2.1). Thus, $L \in \mathfrak{O}$ and consequently, $N = f(L) \in \mathfrak{O}$.

Subcase 1.2. General case for N. The previous argument applied to the group $G/c(N)$ in place of G, and its subgroup $N/c(N)$, in place of N, gives $N/c(N) \in \mathfrak{O}$. Since $\dim c(N) \leq \dim G < \infty$, we conclude that $N \in \mathfrak{O}_c$.

Case 2. General case for G. The closed subgroup $N_1 := N \cap c(G)$ of N obviously contains $c(N)$. Moreover, $N/N_1 \cong (N + c(G))/c(G)$ is isomorphic to a closed subgroup of $G/c(G) \in \mathfrak{O}$. Therefore, $N/N_1 \in \mathfrak{O}$. On the other hand, $N_1/c(N) \in \mathfrak{O}$ as a closed totally disconnected subgroup of $c(G)/c(N)$, by Subcase 1.1. Hence, the totally disconnected group $N/c(N)$ is an extension of two Orsatti groups, so $N/c(N) \in \mathfrak{O}$. Therefore, $N \in \mathfrak{O}_c$. $\square$

We can briefly resume subcase 1.1 as follows: a closed totally disconnected subgroup of a finite-dimensional connected compact abelian group is an Orsatti group.

Theorem 6.4. *A compact abelian group G belongs to $\mathfrak{O}_c$ iff there exists a closed subgroup $N \in \mathfrak{O}$ of G such that G/N is connected with* $\dim G/N < \infty$

Proof. The sufficiency follows from the stability of $\mathfrak{O}_c$ under taking extensions. Assume that $G \in \mathfrak{O}_c$ and present $G = c(G) + N$, where N is a closed totally disconnected subgroup of G. Then $G/c(G) \cong N/(c(G) \cap N) \in \mathfrak{O}$. On the other hand, $N \cap c(G) \in \mathfrak{O}$ by Sub case 1.1 of the above proof. Since $\mathfrak{O}$ is stable under extension and $N \cap c(G)$, as well as $N/(c(G) \cap N)$ belong to $\mathfrak{O}$, we deduce that $N \in \mathfrak{O}$. $\square$

The next theorem describes the compact abelian groups $G \in \mathsf{S}(\mathfrak{E}_{<\infty})$ and shows that they coincide with the compact abelian groups $G \in \mathsf{Q}(\mathfrak{E}_{<\infty})$:

Theorem 6.5. *For every compact abelian group G the following statements are equivalent:*

(i) $G \in \mathfrak{O}_c$.

(ii) G *is finite-dimensional and* $G/pG \in \mathfrak{E}_{<\infty}$ *for every prime* p;

(iii) $G \in \mathsf{Q}(\mathfrak{E}_{<\infty})$, *i.e., every Hausdorff quotient of* G *belongs to* $\mathfrak{E}_{<\infty}$.

(iv) $G \in \mathsf{S}(\mathfrak{E}_{<\infty})$, *i.e., every closed subgroup of* G *belongs to* $\mathfrak{E}_{<\infty}$.

In case these conditions hold, G is metrizable.

Proof. The implication (iii) $\Rightarrow$ (ii) easily follows from Theorem B. The implication (ii) $\Rightarrow$ (i) follows from Example 5.2(a) and the equivalence (a) $\leftrightarrow$ (b) in Proposition 6.2, since $G/pG \cong \mathbb{Z}(p)^\kappa$ for $\kappa = w(G/pG)$.

Since the class $\mathfrak{O}_c$ is stable under taking quotients and closed subgroups, to check the implications $(i) \Rightarrow$ (iii) and $(i) \Rightarrow$ (iv) it suffices to notice that the class $\mathfrak{O}_c$ is contained in $\mathfrak{E}_{<\infty}$. This follows from Theorem B, Proposition 3.4 and Proposition 3.12.

To prove the implication (iv) $\Rightarrow$ (i) we show first that a totally disconnected compact group N satisfying (iv) must be an Orsatti group. Indeed, as the class $\mathfrak{O}$ is defined locally, so we can assume without loss of generality that N is a pro-p-group for some prime p. Our hypothesis gives $N[p] \in \mathfrak{E}_{<\infty}$. As $N[p] \cong \mathbb{Z}(p)^\sigma$ for some σ, $N[p]$ is finite by Example 5.1. Therefore the torsion subgroup of N is finite (as N is reduced). Hence there exists n such that $N[p^n]$ coincides with the torsion subgroup of N. Since the closed subgroup p^nN of G is obviously torsion-free, it must be isomorphic to $\mathbb{Z}_p^\kappa$ for some κ. In other words, $N/N[p^n] \cong \mathbb{Z}_p^\kappa$ and consequently $N \cong N[p^n] \times \mathbb{Z}_p^\kappa$. Now $N \in \mathfrak{E}_{<\infty}$ implies $\kappa < \omega$, hence $N \in \mathfrak{O}$.

Assume (iv) holds true and pick a totally disconnected subgroup N of H such that $G = c(G) + N$. Then $N \in \mathfrak{O}$ by the above argument. Hence, $G/c(G) \cong N/(N \cap c(G)) \in \mathfrak{O}$. This proves (iv) $\Rightarrow$ (i). $\square$

Here we specify Theorem 6.5 in the totally disconnected case (so taking $c(G) = \{0\}$ in that theorem), collecting all equivalent conditions from Theorem 6.5 and Proposition 6.2 in order to obtain a description of $\mathsf{S}(\mathfrak{E}_0)$ and $\mathsf{Q}(\mathfrak{E}_0)$:

Corollary 6.6. *For every totally disconnected compact abelian group G the following statements are equivalent:*

(a) $G \in \mathsf{Q}(\mathfrak{E}_{<\infty})$, *i.e., every quotient of* G *belongs to* $\mathfrak{E}_{<\infty}$;

(b) $G \in \mathsf{Q}(\mathfrak{E}_0)$*, i.e., every quotient of* G *belongs to* $\mathfrak{E}_0$*;*

(c) $G/pG \in \mathfrak{E}_0$ *(equivalently,* $G/pG \in \mathfrak{E}_{<\infty}$*) for every prime* p*;*

(d) $G \in \mathfrak{D}$*.*

(e) $G \in \mathsf{S}(\mathfrak{E}_0)$*, i.e., every closed subgroup of* G *belongs* $\mathfrak{E}_0$*;*

(f) $G[p] \in \mathfrak{E}_0$ *(equivalently,* $G[p] \in \mathfrak{E}_{<\infty}$*) for every prime* p*.*

In particular, $\mathsf{S}(\mathfrak{E}_0) \cap \{\textit{compact abelian groups}\} = \mathsf{Q}(\mathfrak{E}_0) \cap \{\textit{compact abelian groups}\} = \mathfrak{D}$*.*

6.2 Non-abelian case: characterization of dimension via stabilities of $\mathfrak{E}_{<\infty}$

The next theorem, covering item (b_1) of Theorem F, characterizes the finite-dimensional compact connected groups as the compact connected groups having all closed subgroups in $\mathfrak{E}_{<\infty}$, i.e., as the intersection $\{\textit{compact connected groups}\} \cap \mathsf{S}(\mathfrak{E}_{<\infty})$.

Theorem 6.7. *For every compact connected group* G *the following statements are equivalent:*

(i) $\dim G < \infty$*;*

(ii) $G \in \mathsf{S}(\mathfrak{E}_{<\infty})$*, i.e., every closed subgroup of* G *belongs to* $\mathfrak{E}_{<\infty}$*.*

In case these conditions hold, G *is metrizable and* $G \in \mathsf{Q}(\mathfrak{E}_{<\infty})$*.*

Proof. (i) $\Rightarrow$ (ii) Let N be a closed subgroup of G. We need to prove that $N \in \mathfrak{E}_{<\infty}$.

According to Fact 2.4, our hypothesis $\dim G' < \infty$ (a consequence of $\dim G < \infty$), implies that G' is a Lie group. Hence, its closed subgroup $N_0 := N \cap G'$ is a Lie group as well. Let us note that $N/N_0 \in \mathfrak{D}_c$ as N/N_0 is isomorphic to a subgroup of the compact connected finite-dimensional abelian group $G/G' \in \mathfrak{D}_c$ and $\mathfrak{D}_c$ is stable under taking closed subgroups, by Lemma 6.3.

The subgroup $\lambda(N)$ of N is fully invariant and $\lambda(N) \in \mathfrak{E}_{<\infty}$, by Lemma 4.7. So, to get $N \in \mathfrak{E}_{<\infty}$ it suffices to check that $N_1 := N/\lambda(N) \in \mathfrak{E}_{<\infty}$, according to Proposition 3.12(c).

To check that $N_1 \in \mathfrak{E}_{<\infty}$ put $L := N_0\lambda(N)$ and consider the normal subgroup $F := L/\lambda(N) \cong N_0/(N_0 \cap \lambda(N))$ of N_1. Since L is a compact Lie group, the subgroup $c(L) = \lambda(N)$ is open with $[L : \lambda(N)] < \infty$. Therefore, F is finite.

On the other hand, $N_1/F \cong N/L$ is isomorphic to a quotient of the group $N/N_0 \in \mathfrak{D}_c$, hence $N_1/F \in \mathfrak{D}_c$. In particular, N_1/F is abelian, so F contains the commutator subgroup N_1' of N_1.

The finite subgroup N_1' of N_1 is fully invariant and $N_1' \in \mathfrak{E}_{<\infty}$. By Proposition 3.12(b), this yields $N_1 \in \mathfrak{E}_{<\infty}$, provided we check that $N_1/N_1' \in \mathfrak{E}_{<\infty}$. The compact abelian group N_1/N_1' has a finite subgroup F/N_1' such that $(N_1/N_1')/(F/N_1') \cong N_1/F \in \mathfrak{D}_c$, as mentioned above. Hence, $N_1/N_1' \in \mathfrak{D}_c$ and consequently, $N_1/N_1' \in \mathfrak{E}_{<\infty}$. This proves that $N_1 \in \mathfrak{E}_{<\infty}$.

To prove the implication (ii) $\Rightarrow$ (i) it suffices to check that $\dim Z(G) < \infty$ and $\dim G' < \infty$, in view of (3). The former inequality follows from $Z(G) \in \mathfrak{E}_{<\infty}$ (as $Z(G)$ is a closed subgroup of $G \in \mathsf{S}(\mathfrak{E}_{<\infty})$) and Theorem A.

To prove the inequality $\dim G' < \infty$, argue by contradiction. According to Remark 2.5, the hypothesis $\dim G' = \infty$ provides an infinite product $\prod_{i \in I} L_i$ of compact connected simple Lie groups L_i such that $G'/Z(G') \cong \prod_{i \in I} L_i$. For every $i \in I$ let $S_i := \widetilde{L}_i$ be the covering group of L_i and let $T_i \cong \mathbb{T}$ be a closed subgroup of S_i. Then the subgroup $T = \prod_{i \in I} T_i$ of $L = \prod_{i \in I} S_i$ is isomorphic to

$\mathbb{T}^I$. So $\dim T^I = \infty$. By Fact 2.4, there is a closed central subgroup N of L with $\dim N = 0$ such that $G' \cong L/N$, let $f : L \to G'$ be the quotient map and put $C = f(T)$. As $\dim N = 0$, we conclude, with Fact 2.2, that $\dim C = \dim T = \infty$, so, $C \notin \mathfrak{E}_{<\infty}$, by Theorem A. As C is a closed subgroup of G, we deduce that $G \notin \mathfrak{E}_{<\infty}$, a contradiction. $\square$

The above proof combined with lemmas 2.4, 4.6 and Remark 4.3(b), shows also that $G \in \mathsf{Q}(\mathfrak{E}_0)$ for a compact connected group G if and only if $G = G'$ is finite-dimensional.

Example 6.8 shows that the implication in the last assertion of Theorem 6.7 cannot be inverted. More precisely, $\mathfrak{E}_0$ contains an infinite-dimensional compact semi-simple connected group G along with all quotients of G. In terms of the theorem, this means that $G \in \mathsf{Q}(\mathfrak{E}_0)$, yet $G \notin \mathsf{S}(\mathfrak{E}_{<\infty})$. This should be compared the Theorem 6.5 establishing coincidence of $\mathsf{S}(\mathfrak{E}_{<\infty})$ and $\mathsf{Q}(\mathfrak{E}_{<\infty})$ in the realm of compact abelian groups.

Example 6.8. Take $G = \prod_{n=1}^{\infty} L_n$, where L_n are pairwise non-isomorphic compact connected simple Lie groups. Then G is a compact infinite-dimensional connected group. We show that every continuous endomorphism of G has zero topological entropy, i.e., $G \in \mathfrak{E}_0$.

(a) For simplicity, we take $L_n = SO_{2n+1}(\mathbb{R})$ for $n \geq 1$ (see item (b) for the general case). Assume $f : G \to G$ is a continuous endomorphism. In case f is not surjective, f need not be a product of continuous endomorphisms of each component L_n (in the surjective case, one necessarily has $f(L_n) = L_{m_n}$ for some m_n, so necessarily $m_n = n$ in this case, see [13] for more detail). Let

$$N_m := \prod_{n=1}^{m} L_n, \quad G_m = \prod_{n=m+1}^{\infty} L_n \quad \text{and} \quad S_m = \bigoplus_{n=m+1}^{\infty} L_n, \tag{10}$$

so that $G = N_m \times G_m$ and $G_m = \overline{S}_m$ for every $m \in \mathbb{N}$. In the sequel we identify, when necessary, L_n with the respective coordinate subgroup in the product G. Using the fact that $\dim L_i < \dim L_n$ for all every $i < n$, we conclude that natural projection $p_i : G \to L_i$ sends $f(L_n)$ to $\{e\}$, since $f(L_n)$ is either trivial, or isomorphic to L_n. Thus, $f(L_m) \leq G_n$ for all $m > n$. Hence, $f(S_m) \leq G_m$ and consequently $f(G_m) \leq G_m$, as $G_m = \overline{S}_m$. Denote by f_m the endomorphism of $N_m \cong G/G_m$ induced by f (notice that this makes sense as G_m is f-invariant, even if N_m need not be f-invariant, i.e., f_m is not a restriction of f in general). Denoting by $\varphi_m : N_{m+1} \to N_m$ the natural projection, we obtain an inverse system (N_m, φ_m) with inverse limit $G \cong \varprojlim N_m$ such that the canonical projections $\pi_m : G \to N_m$ satisfy $f_m \circ \pi_m = \pi_m \circ f$, so $f \cong \varprojlim f_m$. By Fact 3.7, we have $h(f) = \lim_m h(f_m)$. To conclude, it suffices to note that $h(f_m) = 0$, by Remark 4.3 (b), as each H_m is a semisimple Lie group. Therefore, $h(f) = 0$.

(b) In the general case the proof follows the line of the argument from item (a), using the fact that for every $m \in \mathbb{N}$ there are at most finitely many groups L_n with $\dim L_n \leq m$. Following this line, define the counterpart of (10), namely the subgroups

$$N_m = \prod\{L_n : \dim L_n \leq m\} \quad \text{and} \quad G_m = \prod\{L_n : \dim L_n > m\},$$

with $G = N_m \times G_m$ and $f(G_m) \leq G_m$ as above. The proof can be concluded exactly as in (a) (for more detail and for more general results see [13]).

Finally, a standard argument shows that every closed normal subgroup N of G is simply a subproduct $N = \prod_{n \in J} L_n$, $J \subseteq \mathbb{N}$. Hence, the quotient G/N has the form $\prod_{n \in \mathbb{N} \setminus J} L_n$. Therefore, $G/N \in \mathfrak{E}_0$ by the above proof. This proves that $G \in \mathsf{Q}(\mathfrak{E}_0)$.

Now we prove item (b_2) of Theorem F.

Theorem 6.9. *For every compact connected group G the following statements are equivalent:*

(i) $\dim Z(G) < \infty$ *and G has no Lie components of infinite mutiplicity;*

(ii) $G \in \mathsf{Q}(\mathfrak{E}_{<\infty})$, *i.e., every quotient subgroup of G belongs to $\mathfrak{E}_{<\infty}$.*

In case these conditions hold, G is metrizable.

Proof. For the sake of brevity, put $A = c(Z(G))$. To prove the implication (ii) $\Rightarrow$ (i) deduce from the equality $G = A \cdot G'$ that

$$G/G' \cong A/(A \cap G') \in \mathsf{Q}(\mathfrak{E}_{<\infty}). \tag{11}$$

So (11) implies $\dim Z(G) = \dim A = \dim A/(A \cap G') < \infty$ in view of $\dim A \cap G' = 0$ and Theorem B, as $A/(A \cap G')$ is abelian.

To prove the second assertion, note that $G/Z(G) \in \mathfrak{E}_{<\infty}$. As $G/Z(G) \cong \prod_i L_i$, none of these Lie group L_i has infinite multiplicity by Example 5.1, as otherwise $G/Z(G)$ will have a direct summand that is an infinite power of an infnite compact group

To prove the implication (i) $\Rightarrow$ (ii) assume that $\dim Z(G) < \infty$ and G has no Lie components of infinite multiplicity. We check first that all quotients of G still satisfy (i).

Let G/N be a quotient of G with respect to a closed normal subgroup N of G. First we see that the quotient map $q : G \to G/N$ satisfies

$$Z(G/N) = q(Z(N)). \tag{12}$$

Indeed, as q is surjective, one obviously has $q(Z(G)) \leq Z(G/N)$. In order to prove that these two subgroups coincide consider the surjective canonical homomorphism $f : G/N \to (G/N)/q(Z(G)) =: H$. It suffices to check that the image of $Z(G/N)$ under this homomorphism is trivial. Since this image must be obviously contained in the centre of H, it suffices to see that H is center-free. This follows in turn form the fact that H is isomorphic to the quotient of the group $G/Z(G)$ with respect to its normal subgroup $L := (Z(G)N)/Z(G)$. According to Fact 2.4(b), $G/Z(G) \cong \prod_{i\in I} S_i$ for a family of simple center-free connected Lie groups S_i. Then the normal subgroup L of $G/Z(G)$ will correspond, under this isomorphism, to a subproduct $\prod_{i\in I'} S_i$. Consequently,

$$H = (G/N)/q(Z(G)) \cong G/Z(G)N \cong (G/Z(G))/L$$

is isomorphic to the quotient $\prod_{i\in I} S_i/\prod_{i\in I'} S_i \cong \prod_{i\in I\setminus I'} S_i$. Therefore, H has trivial center. This proves (12).

The equality (12) and the monotonicity of dimension imply $\dim Z(G/N) < \infty$. To prove that G/N satisfies also the second half of (i) note that by virtue of (12), q induces a surjective homomorphism $G/Z(G) \to (G/N)/Z(G/N)$. Since these groups are products of centre-free groups, the components of $(G/N)/Z(G/N)$ appear as components of $G/Z(G)$. In particular, G/N has no Lie components of infinite multilpicity.

We proved in this way that if G satisfies (i), then all quotients of G satisfy (i). Hence, it remains to see that a group G satisfying (i) belongs to $\mathfrak{E}_{<\infty}$.

To this end we prove first that $G' \in \mathfrak{E}_0$. Its center $Z(G') = A \cap G'$ is a characteristic subgroup and $Z(G') \in \mathfrak{E}_0$, as $Z(G')$ coincides with the zero-dimensional (by Fact 2.4(b)) subgroup $A \cap G'$ of the finite dimensional connected abelian group A. So, by Proposition 3.12 it remains to see that $G'/Z(G') \in \mathfrak{E}_0$. Since $G'/Z(G')$ is centre-free, it is isomorphic to a product $\prod_i L_i$ of simple Lie groups (by Fact 2.4(b)).

Moreover, by the second part of our hypothesis each Lie group appears with finite multiplicity in this product (in particular, the product is countable). Now Example 6.8 (b) implies that $G'/Z(G') \in \mathfrak{E}_0$.

Hence, our hypothesis $\dim Z(G) < \infty$ yields $\dim G/G' < \infty$, as $\dim G/G' = \dim Z(G)$, by (3). According to Theorem B, $G/G' \in \mathfrak{E}_{<\infty}$, since G/G' is abelian. As $G' \in \mathfrak{E}_0$ is fully invariant, this yields $\mathfrak{E}_{<\infty}$, by Proposition 3.12(b). This proves the implication (i) $\Rightarrow$ (ii).

Finally, we note that a group G satisfying (i) is metrizable. Indeed, as $G = A \cdot G'$ it suffices to check that both A and G' are metrizable. The former group is metrizable, being a connected finite-dimensional abelian group. To see that G' is metrizable, it suffices to note that G' is an extension of the metrizable normal subrgroup $Z(G') = A \cap G'$ by the group $G'/Z(G') \cong \prod_i L_i$. This product of Lie groups is metrizable, as the second hypothesis in (i) yields that the product is countable. $\Box$

An easy modification of the above proof allows us to deduce that $G \in \mathsf{Q}(\mathfrak{E}_0)$ for a compact connected group G if and only if $G = G'$ and has no Lie components of infinite multiplicity. Hence, the class $\mathsf{Q}(\mathfrak{E}_0) \cap \{\text{connected compact groups}\}$ properly contains $\mathsf{S}(\mathfrak{E}_0) \cap \{\text{connected compact groups}\}$, unlike the case of abelian groups (see Corollary 6.6).

7 Final remarks and open questions

One can replace endomorphisms by automorphisms and consider the bigger classes $\mathfrak{A}_{<\infty}$ and $\mathfrak{A}_0$ defined with respect to automorphisms, namely:

- $\mathfrak{A}_{<\infty}$ – the class of topological groups that have no automorphism of infinite entropy; and
- $\mathfrak{A}_0$ – the class of topological groups G that such that every automorphism of G has zero entropy.

Now the use of the endomorphisms m_k (with $k > 1$) has a limited range, since they need not be automorphisms in general. Obviously, $\mathfrak{E}_{<\infty} \subseteq \mathfrak{A}_{<\infty}$ and $\mathfrak{E}_0 \subseteq \mathfrak{A}_0$. Easy examples show that these inclusions, as well as the inclusion $\mathfrak{A}_0 \subseteq \mathfrak{A}_{<\infty}$, are proper.

We see in the next example that $\mathfrak{E}_{<\infty} \not\supseteq \mathfrak{A}_0$.

Example 7.1. An abelian group G is called self-rigid, if $End(G) = \{m_G^k : k \in \mathbb{Z}\}$. Clearly, $Aut(G) = \{\pm id_G\}$ for a self-rigid group G, thus the only continuous automorphisms of $K = \widehat{G}$ are $\pm id_K$. Therefore, $K \in \mathfrak{A}_0$. On the other hand, self-rigid groups of arbitrarily large size were built by De Groot, Fuchs and Shelah. In particular, if G is an uncountable self-rigid group, then G has infinite rank, so $K = \widehat{G}$ is an infinite-dimensional compact connected abelian group. Thus, $\mathbf{E}_{top}(K) = \{0, \infty\}$, in virtue of Theorem A. Therefore, $K \notin \mathfrak{E}_{<\infty}$, while $K \in \mathfrak{A}_0$.

Question 7.2. Describe the classes $\mathfrak{A}_0$ and $\mathfrak{A}_{<\infty}$ within

(i) the class of all compact abelian groups;

(ii) the class of all compact connected groups.

Question 7.3. Does $G \in \mathfrak{E}_{<\infty}$ for a compact abelian group G imply $G/c(G) \in \mathfrak{E}_{<\infty}$?

Item (a) of Theorem B along with a positive answer to Question 7.3 will completely reduce the study of the compact abelian groups in the class $\mathfrak{E}_{<\infty}$ to those compact abelian groups of $\mathfrak{E}_{<\infty}$ that are totally disconnected.

Using the Bridge Theorem one may try to negatively answer this question by looking for a (discrete) abelian group X, with $X \in \mathfrak{E}_{<\infty}$, but $t(X) \notin \mathfrak{E}_{<\infty}$. A *similar* example to the effects of the entropy ent was given in [11, Example 5.5]. Namely, a group X such that $t(X) \notin \mathfrak{E}_{<\infty}$ and every endomorphism f of X has $\mathrm{ent}(f) = 0$. Unfortunately, this example does not fit our question, since m_X^k has infinite algebraic entropy, as the free-rank of X is $\mathfrak{c}$.

We are not aware if the condition $G \in \mathfrak{E}_{<\infty}$ alone for a compact connected abelian group G may imply any of the two properties in item (a) of Theorem 6.9.

The general problem of when a compact abelian group belongs to $\mathfrak{E}_{<\infty}$ or $\mathfrak{A}_{<\infty}$ remains open.

Question 7.4. Are the classes (of compact abelian groups in) $\mathfrak{E}_{<\infty}$ and $\mathfrak{E}_0$ closed with respect to:

(i) taking finite products;

(ii) taking extensions?

We do not know whether either of the two conditions in item (i) of Theorem 6.9 is necessary for a connected compact group G to satisfy $G \in \mathfrak{E}_{<\infty}$. In other words:

Question 7.5. Does there exist a connected compact group $G \in \mathfrak{E}_{<\infty}$ such that either $\dim Z(G) = \infty$ or G has some Lie component of infinite mutilpicity?

Note that a negative answer to this question will imply that for connected compact groups $G \in \mathfrak{E}_{<\infty}$ is equivalent to $G \in \mathsf{Q}(\mathfrak{E}_{<\infty})$ (in striking contrast with the abelian case Corollary 6.6).

We believe that the following, somewhat weaker and "asymmetric", conjecture holds true:

Conjecture 7.6. *If a connected compact group G to satisfies $\dim A < \infty$, then $G \in \mathfrak{E}_{<\infty}$ if and only if every Lie component of G has finite multiplicity.*

Actually, modulo Theorem 6.9, what is missing is only a proof that if $\dim A < \infty$ and $G \in \mathfrak{E}_{<\infty}$ then every Lie component of G has finite multiplicity.

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